Siegel Modular Varieties and the Eisenstein Cohomology of PGL_{2g+1}

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Abstract

We use the twisted topological trace formula developed in [Wes12] to understand liftings from symplectic to general linear groups. We analyse the lift from Sp_{2g} to PGL_{2g+1} over the ground field $\mathbb{Q}$ in further detail, and we get a description of the image of this lift of the L^2 cohomology of Sp_{2g} (which is related to the intersection cohomology of the Shimura variety attached to GSp_{2g}) in terms of the Eisenstein cohomology of the general linear group, whose building constituents are cuspidal representations of Levi groups. This description may be used to understand endoscopic and CAP-representations of the symplectic group.

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1 Introduction

This paper is a sequel of our paper [Wes12], where we developed a twisted topological trace formula and tried to understand liftings from symplectic to general linear groups. Here we want to analyse the lift from Sp_{2g} to PGL_{2g+1} over the ground field $\mathbb{Q}$ in further detail, and we get a description of the image of this lift for the L^2 cohomology of Sp_{2g} (which is related to the intersection cohomology of the Shimura variety attached to GSp_{2g}) in terms of the Eisenstein cohomology of the general linear group, whose building constituents are cuspidal representations of Levi groups. This description may be used to understand endoscopic and CAP-representations of the symplectic group. Roughly speaking we do not classify cohomology classes for the symplectic group with respect to their cuspidal support on Sp_{2g} in the sense of [FS98] but in terms of their cuspidal support on their lifts to PGL_{2g+1}. In view of the strong multiplicity one theorem on GL_{2g+1} this gives a classification of all "cohomological" packets (equivalence classes of automorphic representations which are isomorphic at all but finitely many places) in the discrete spectrum of Sp_{2g}.

Let us describe what we will do in more detail, starting with the heart of our arguments:

In chapter 9 we use the fundamental work of Franke [Fr98] to write the cohomology of G as a sum of parabolically induced L^2 cohomologies attached to the different Levi subgroups of G and with respect to suitable coefficient systems on these Levis. For $G = \mathrm{PGL}_{2g+1}$ the index sum may be rewritten in terms of partitions of the set $J = \{-g, \ldots, g\}$. In chapter 10 we compare these sums for the goups in question using an induction procedure and get that the L^2-cohomology attached to $G = \mathrm{Sp}_{2g}$ lifts to a sum, indexed by η-invariant partitions of J, of the parabollically induced L^2-cohomology attached to suitable Levi factors (which are products of general linear groups) and to suitable coefficient systems (Theorem 10.15). Here a partition $J = \bigcup_{i=1}^{r} J_i$ is called η-invariant if we have $-J_i = J_i$ for $i = 1, \ldots, r$.

Now one can go further: the discrete spectrum of general linear groups is well understood by the work of Moeglin and Waldspurger [MW89]: each automorpic representation in the discrete spectrum is a Langlands quotient $MW(\pi^{(i)}, n_i)$ of some parabollically induced representation, in which a cuspidal $\pi^{(i)}$ appears n_i times with various twists by powers of the modulus character. The extended version of the strong multiplicity for GL_n ([JS81b]) then implies that an η-invariant automorphic representation contributing to the lifted cohomology on PGL_{2g+1} comes from η-invariant cuspidal representations $\pi^{(i)}$ on some smaller linear groups $G^{(i)}$. By the work of Cogdell, Kim, Piatetski-Shapiro, Shahidi and Soudry ([CKPSS01], [CKPSS04], [Sou05], [CPSS11]) one knows that $\pi^{(i)}$ (resp. the twist of $\pi^{(i)}$ by a quadratic character in the case $G^{(i)} = \mathrm{GL}_{2\gamma_i+1}$) is a semi-weak lift of a globally generic cuspidal representation $\pi_1^{(i)}$ on some quasi-split classical group $G_1^{(i)}$.

In case $G^{(i)} = \mathrm{GL}_{2\gamma_i+1}$ one knows that $G_1^{(i)} = \mathrm{Sp}_{2\gamma_i}$. In case $G^{(i)} = \mathrm{GL}_{2\gamma_i}$ one

knows that $G_1^{(i)} = \mathrm{SO}_{2\gamma_i+1}$ if $L(s,\pi^{(i)},\Lambda^2)$ has a pole at $s=1$, but $G_1^{(i)} = \mathrm{SO}^d_{2\gamma_i}$ if $L(s,\pi^{(i)},Sym^2)$ has a pole at $s=1$. Here the discriminant of the quadratic form describing the quasi split $\mathrm{SO}^d_{2\gamma_i}$ is related via class field theory to the central character ω_i of $\pi^{(i)}$, which is quadratic. If $\omega_i \neq 1$ we are always in the case of an even orthogonal group. But since the notion of a semi-weak lift means that the lift is compatible with the local Langlands isomorphism at the archimedean prime, we do not have to look for poles of L-functions: the fact that $\pi^{(i)}$ is cuspidal implies that $\pi^{(i)}_\infty$ is tempered and then the Langlands data are completely determined by the coefficient system. But by lemma 4.2 already the coefficient system decides, whether we are in the odd or in the even orthogonal case.

If one considers this paper as a contribution to "Langlands-functoriality for the cohomology of arithmetic groups", one has to think about the question, what Langlands-functoriality for coefficient systems V_χ should mean. Here V_χ is the highest weight module for some dominant weight $\chi \in X^*(T)$ on a maximal torus $T \subset G$. We will see, that $\chi + \delta_G \in X^*(T) \otimes \mathbb{C} = X_*(T) \otimes \hat{T}$ behaves functorial, where δ_G is as usual the half sum of the positive roots and where $\hat{T}$ is a maximal torus in the dual group: The action of the center of the universal enveloping algebra $\mathcal{Z}(\mathfrak{g})$ on V_χ is given by $\chi + \delta_G$, and by Wigner's lemma this is up to sign the action of $\mathcal{Z}(\mathfrak{g})$ on the real representations π_∞ contributing to cohomology. Now this action is related to the Langlands parameters of π_∞, which behave functorial. In the case of GL_{2g+1} this means that an η-invariant coefficient system is already described by the (characteristic) set S of all (integral) entries of the vector $\chi + \delta_G \in X^*(T) = \mathbb{Z}^{2g+1}$, since these entries form a strictly increasing sequence. Here η-invariance means $-S = S$.

In order to include the construction of Mœglin-Waldspurger to describe the discrete spectrum of general linear groups into the notations we define characteristic sets in chapter 2 as finite non empty subsets S of $\frac{1}{2}\mathbb{Z}$, such that $-S = S$ and $s - t \in \mathbb{Z}$ for all $s,t \in S$. These are of type C_g if $0 \in S$, of type B_g if $S \subset \frac{1}{2} + \mathbb{Z}$ and finally of type D_g if $S \subset \mathbb{Z}$, but $0 \notin S$. For $n \geq 1$ and a characteristic set S we define $MW(S,n) = \{s - \frac{n+1}{2} + i | 1 \leq i \leq n\}$ if S is n-admissible in the sense of definition 2.1(b).

After this short set theoretic beginning we recall in chapter 3 the notion of (stable) twisted endoscopy, introduce the groups involved in the game and relate the coefficient systems to characteristic sets. In chapter 4 we analyse the real representations contributing to cohomology for a given coefficient system and their Langlands parameters. Furthermore we calculate the Lefschetz numbers for the η-action on the cohomology of the real representations.

In chapter 5 we formulate the main theorem about the structure of lifts (Theorem 5.3): To an irreducible automorphic representation τ of $\mathrm{Sp}_{2g}(\mathbb{A})$ in the discrete spectrum, which is cohomological with respect to some coefficent system V_χ with characteristic set $S = S_\chi$ one associates a family of octupels:

$$(X_{\gamma_i}, G^{(i)}, G_1^{(i)}, S_i, n_i, d_i, \pi^{(i)}, \pi_1^{(i)})_{i=1,\dots,r},$$

which describes the lift of τ to PGL_{2g+1} in a way we already discussed. One should not be afraid of this description by octupels: to get all possible such families one should at first look for partitions of S into integral characteristic sets: $S = \bigcup_{i=1}^r \tilde{S}_i$. One of them is of type C_{γ_i}, the others are of type B_{γ_i}.

Then one should look for integers n_i such that $\tilde{S}_i = MW(S_i, n_i)$ where S_i is an n_i-admissible characteristic set. Observe that for every i one always can take $n_i = 1$ since $MW(S, 1) = S$. Let X_{γ_i} be the type of S_i, and let $d_i \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ be a family of square classes such that $d_i = 1$ for type B_{γ_i}, $(-1)^{\gamma_i} d_i > 0$ for type D_{γ_i} and $\Pi_{i=1}^r d_i = 1$. Then the S_i and d_i determine the classical groups $G_1^{(i)}$ and the linear groups $G^{(i)}$ in question. Finally $(\pi^{(i)}, \pi_1^{(i)})$ is a pair of irreducible globally generic cuspidal automorphic representations for these two groups, which have the correct Langlands parameters at the archimedean prime and such that $\pi^{(i)}$ has central character χ_{d_i} and is a semi weak lift of $\pi_1^{(i)}$. The proof of the main theorem and its converse is given in the later chapter 11 and includes the arguments already sketched.

In the last chapter 12 we describe the elementary pairs $(\pi^{(i)}, \pi_1^{(i)})$ in the case of characteristic sets of type C_1, D_1, B_1 and D_2, we analyse the case $g = 2$ in further detail, and give hints to understand the weights appearing in the recent work of Bergström, Faber and van der Geer [BFG11] in case $g = 3$, explain how the Ikeda lift fills into our language and finally try to give a relation to elliptic endoscopic groups for GSp_{2g} as appearing in the work of Morel [Mor11].

Finally we have to mention some technical computations: We will calculate and compare the factors $\alpha_\infty(\gamma_0, h_\infty)$ from the topological trace formula (restated in chapter 8) in chapter 7, after we carried out in chapter 6 a comparison of volumes of the real compact forms of Sp_{2g} and SO_{2g+1} with respect to measures attached to Chevalley basis. This will make the comparison of topological trace formulas more explicit, and we get a more precise meaning of the statement that the cohomology attached to PGL_{2g+1} is a lift of the cohomology attached to Sp_{2g}. Here we mean by "cohomology attached to an algebraic group G" the alternating sum of the cohomologies of locally symmetric spaces attached to G with coefficients in some local system coming from a finite dimensional algebraic representation V_χ of G.

It should be noted that their may be some overlap of our work with the forthcoming book of Arthur [Ar12]: His results depend on the stabilization of the twisted analytic trace formula, which seems not to be established yet, but give much more precise statements about the local and global packets of (automorphic) representations of $\mathrm{Sp}_{2g}(\mathbb{A})$. But for applications to cohomology and for a first understanding of the lift to PGL_{2g+1} our approach seems to be more direct.

2 Characteristic sets

Definition 2.1. *(a) A characteristic set is a finite subset $S \neq \emptyset$ of $\frac{1}{2}\mathbb{Z}$, such that $-s \in S$ for all $s \in S$ and $s - t \in \mathbb{Z}$ for all $s, t \in \mathbb{Z}$*

(b) For $n \in \mathbb{N}$ a characteristic set is called n-admissible if we have $|s-t| \geq n$ for all $s, t \in S$ with $s \neq t$.

(c) For $n \in \mathbb{N}$ we call $E_n = \left\{-\frac{n+1}{2} + i \mid 1 \leq i \leq n\right\}$ the elementary characteristic set of order n.

(d) If S is an n-admissible characteristic set we put

$$MW(S,n) = \{s + e \mid s \in S, e \in E_n\}.$$

Lemma 2.2. *Let $n \in \mathbb{N}$ and S be an n-admissible characteristic set.*

(a) Each E_n and each $MW(S,n)$ is a characteristic set.

(b) The map $\alpha : S \times E_n \to MW(S,n),\ (s,e) \mapsto s + e$ is a bijection.

□

Remark 2.3. It is obvious that each characteristic set S is of exactly one of the following types for some $g \in \mathbb{N}$:

(a) S is of type B_g $(g \geq 1)$, if $S \subset \mathbb{Z} + \frac{1}{2}$ and $\#(S) = 2g$;

(b) S is of type C_g $(g \geq 0)$, if $0 \in S$ and $\#(S) = 2g + 1$;

(c) S is of type D_g $(g \geq 1)$, if $S \subset \mathbb{Z}, 0 \notin S$ and $\#(S) = 2g$.

The characteristic sets of type C_g and D_g are called *integral*. Let S be an n-admissible characteristic set. If n is odd, then $MW(S,n)$ is of the same type (but with different index) as S. If $n > 0$ is an even integer, then $MW(S,n)$ is integral if and only if S is not integral.

3 Endoscopic groups and coefficient modules

(3.1) SPLIT GROUPS WITH AUTOMORPHISM. Let G/F be a connected reductive split group over a field F, and let $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ be a splitting, i.e. T is a maximal split torus inside an F-rational Borel B, $\Delta = \Delta_G = \Delta(G, B, T) \subset \Phi(G, T) \subset X^*(T) = Hom(T, \mathbb{G}_m)$ denotes the set of simple roots inside the system of roots, and the X_α for the simple roots $\alpha \in \Delta$ are a system (nailing) of generators of the root spaces $\mathfrak{g}_\alpha$ in the Lie algebra. Let $\eta \in Aut(G)$ be an automorphism of G of finite order l, which fixes the splitting. We denote by $\tilde{G} = G \rtimes \langle \eta \rangle$ the (nonconnected) semidirect

product of G with η. η acts on the cocharacter module $X_*(T) = Hom(\mathbb{G}_m, T)$ via $X_*(T) \ni \alpha^\vee \mapsto \eta \circ \alpha^\vee$ and on $X^*(T)$ via $\alpha \mapsto \alpha \circ \eta^{-1}$.

(3.2) THE DUAL GROUP. Let $\hat{G} = \hat{G}(\mathbb{C})$ be the dual group of G. By definition $\hat{G}$ has a splitting $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ such that we have identifications $X^*(\hat{T}) = X_*(T)$, $X_*(\hat{T}) = X^*(T)$, which identify the (simple) roots $\hat{\alpha} \in X^*(\hat{T})$ with the (simple) coroots $\alpha^\vee \in X_*(T)$ and the (simple) coroots $\hat{\alpha}^\vee \in X_*(\hat{T})$ with the (simple) roots $\alpha \in X^*(T)$. There exists a unique automorphism $\hat{\eta}$ of $\hat{G}$ which stabilizes $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ and induces on $(X_*(\hat{T}), X^*(\hat{T}))$ the same automorphism as η on $(X^*(T), X_*(T))$.

For some $\hat{s} \in \hat{T}$ let $\hat{H} = (\hat{G}^{\hat{\eta}\hat{s}})^\circ$ be the connected component of the subgroup of elements in $\hat{G}$ which are fixed under $\hat{\eta} \circ int(\hat{s})$. It is a reductive group with a splitting satisfying $\hat{B}_H = \hat{B}^{\hat{\eta}\hat{s}}$, $\hat{T}_H = \hat{T}^{\hat{\eta}}$ and where in the case $\hat{s} = 1$ the family $\{\hat{X}_{\hat{\beta}}\}_{\beta \in \Delta_{\hat{H}}}$ is described in [Wes12, 5.3 and 5.4].

Definition 3.3 (η-endoscopic group). *In the above situation a connected reductive split group H/F will be called an η-endoscopic group for $(G, \eta, \hat{s})$ if its dual group together with the splitting is isomorphic to the above $(\hat{H}, \hat{B}_H, \hat{T}_H, \{X_\beta\}_{\beta \in \Delta_{\hat{H}}})$. In the case $\hat{s} = 1$ we call H a stable η-endoscopic group for (G, η).*

For a maximal split torus $T_H \subset H$ we have:

(i) $\quad X^*(T_H) = X_*(\hat{T}_H) = X_*(\hat{T})^{\hat{\eta}} = X^*(T)^\eta$, and

(ii) $\quad X_*(T_H) = X^*(\hat{T}_H) = (X^*(\hat{T})_{\hat{\eta}})_{free} = (X_*(T)_\eta)_{free}$

is the maximal free abelian quotient of the coinvariant module $X^*(\hat{T})_{\hat{\eta}} = X_*(T)_\eta$.

(3.4) THE CLASSICAL GROUPS. We use the following notations:
$diag(a_1, \ldots, a_n) \in \mathrm{GL}_n$ denotes the diagonal matrix $(\delta_{i,j} \cdot a_i)_{ij}$ and
$antidiag(a_1, \ldots, a_n) \in \mathrm{GL}_n$ the antidiagonal matrix $(\delta_{i,n+1-j} \cdot a_i)_{ij}$. Consider

$$J = J_n = (\delta_{i,n+1-j}(-1)^{i-1})_{1 \le i,j \le n} = antidiag(1, -1, \ldots, (-1)^{n-1}) \in \mathrm{GL}_n(F).$$

and its modification $J'_{2g} = antidiag(1, -1, 1, \ldots, (-1)^{g-1}, (-1)^{g-1}, \ldots, 1, -1, 1)$. Since ${}^tJ_n = (-1)^{n-1} \cdot J_n$ and since J'_{2g} is symmetric we can define the

standard symplectic group	Sp_{2g}	$=$	$\mathrm{Sp}(J_{2g})$
standard split odd orthogonal group	SO_{2g+1}	$=$	$\mathrm{SO}(J_{2g+1})$.
standard split even orthogonal group	SO_{2g}	$=$	$\mathrm{SO}(J'_{2g})$.

We consider the groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2g}, \mathrm{SO}_n$ with the splittings consisting of the diagonal torus, the Borel consisting of upper triangular matrices and the standard nailing. The following map defines an involution of $\mathrm{GL}_n, \mathrm{SL}_n$ and PGL_n:

(iii) $$\eta = \eta_n : g \mapsto J_n \cdot {}^tg^{-1} \cdot J_n^{-1}.$$

Example 3.5 ($A_{2g} \leftrightarrow C_g$)**.** The group $G = \mathrm{PGL}_{2g+1}$ with the involution $\eta = \eta_{2g+1}$ has the dual group $\hat{G} = \mathrm{SL}_{2g+1}(\mathbb{C})$ with involution $\hat{\eta} = \eta_{2g+1}$. Then $H = \mathrm{Sp}_{2g}$ is a stable η-endoscopic group, since its dual is $\hat{H} = \mathrm{SO}_{2g+1}(\mathbb{C}) = \hat{G}^{\hat{\eta}}$.

Example 3.6 ($A_{2g-1} \leftrightarrow B_g$)**.** The group $G = \mathrm{GL}_{2g} \times \mathbb{G}_m$ has the involution $\eta : (\gamma, a) \mapsto (\eta_{2g}(\gamma), \det(\gamma) \cdot a)$ and the dual $\hat{\eta} \in Aut(\hat{G})$ satisfies $\hat{\eta}(c, b) = (\eta_{2g}(c) \cdot b, b)$. Then $\hat{H} = \hat{G}^{\hat{\eta}} = \mathrm{GSp}_{2g}(\mathbb{C})$ is the dual of the stable η-endoscopic group $H = \mathrm{GSpin}_{2g+1} = \left(\mathbb{G}_m \times \mathrm{Spin}_{2g+1}\right)/\mu_2$.

If we consider the involution η_{2g} on the group $G = \mathrm{GL}_{2g}$ we get the stable η-endoscopic group $H = \mathrm{SO}_{2g+1}$ with dual group $\hat{H} = \hat{G}^{\hat{\eta}} = \mathrm{Sp}_{2g}(\mathbb{C})$.

Example 3.7 ($A_{2g-1} \leftrightarrow D_g$)**.** For the group $G = \mathrm{GL}_{2g}$, the involution $\eta := \eta_{2g}$ and the element $\hat{s} = diag(1, \ldots, 1, -1, \ldots, -1) \in \hat{T}$ we get $H = \mathrm{SO}_{2g}$ as an η-endoscopic group, since its dual is $\hat{H} = (\hat{G}^{\hat{\eta}\hat{s}})^\circ = \mathrm{SO}_{2g}(\mathbb{C})$.

For the group $G = \mathrm{GL}_{2g} \times \mathbb{G}_m$, the involution η as in example 3.6 and the element $\hat{s} = (diag(1, \ldots, 1, -1, \ldots, -1); 1) \in \hat{T}$ we get $H = \mathrm{GSpin}_{2g}$ as an η-endoscopic group, since its dual is $\hat{H} = (\hat{G}^{\hat{\eta}\hat{s}})^\circ = \mathrm{GSO}_{2g}(\mathbb{C})$ (see 3.14 below for details).

(3.8) Irreducible finite dimensional representations. The isomorphism classes of finite dimensional algebraic representations of $G/\bar{\mathbb{Q}}$ are in one to one correspondence with the positive Weyl chamber

$$\textbf{(iv)} \qquad X^*(T)^+ \quad = \quad \{\chi \in X^*(T) |\ \langle \chi, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

in such a way that $\chi \in X^*(T)^+$ corresponds to the irreducible representation $V_\chi = (V_\chi, \rho_\chi)$ of highest weight χ. We say that an irreducible finite dimensional representations $\tilde{V}$ of $\tilde{G}$ is η-extended if it remains irreducible after restriction to G and if η acts as identity on the one dimensional space of highest weight vectors. Since we have ${}^\eta V_\chi = V_{\eta(\chi)}$ (where ${}^\eta V_\chi = (V_\chi, \rho_\chi \circ \eta)$) we get a bijection between the isomorphism classes of η-extended irreducible finite dimensional representations $\tilde{V}_\chi$ of $\tilde{G}$ and the set $X^*(T)^+ \cap X^*(T)^\eta$. Denote by $\tilde{V}_\chi$ the η-extended representation which restricts to V_χ.

Let $\rho = \rho_G = \frac{1}{2} \sum_{\alpha > 0} \alpha \in X^*(T) \otimes \mathbb{Q}$ be half the sum of the positive roots.

Lemma 3.9. *Let H be a stable η-endoscopic group for (G, η) and let T_H, T be maximal split tori. Using the identification $X^*(T_H) = X^*(T)^\eta$ we have:*

(a) $X^*(T_H)^+ = X^*(T)^\eta \cap X^*(T)^+$

(b) $\rho_G = \rho_H$

(c) the maps $\chi \mapsto V_{H,\chi}, \quad \chi \mapsto \tilde{V}_\chi$*, give bijections between*

– *the set* $X^*(T_H)^+ = X^*(T)^\eta \cap X^*(T)^+$

– *the isomorphism classes of finite dimensional irr. representations of* H

– *the isom. classes of* η*-extended finite dim. irr. representations of* $\tilde{G}$.

Proof: (a) The simple coroots of H are the images of the coroots of G under the natural projection $p : X_*(T) \mapsto X_*(T_H)$ ([Wes12, Prop. 5.4.]). If we interpret coroots as linear forms on the character group, then p has to be interpreted as the restriction of linear forms from $X^*(T)$ to $X^*(T_H) = X^*(T)^\eta$. In this sense the simple coroots of H are the restrictions of the simple coroots of G, and the claim is a consequence of the definition (**iv**).

(b) Assume without loss of generality that G is simple. Recall ([Wes12, 5.3.]) that the roots $\beta \in \Phi(H, T_H)$ are of the form $\beta = c(\alpha)\cdot S_\eta(\alpha)$, where $S_\eta(\alpha)$ denotes the sum over all elements in the η-orbit of $\alpha \in \Phi(G,T) \subset X^*(T)$ and where $c(\alpha) \in \{1,2\}$. Here α runs through a set of representatives for those η-orbits in $\Phi(G,T)$, whose elements are not "long" in the sense that in the case of a root system of type A_{2g}, $ord(\eta) = 2$ they cannot be written in the form $\alpha_0 = \alpha_1 + \eta(\alpha_1)$ with $\alpha_1 \in \Phi(G,T)$ (call such α_1 short).
We have $c(\alpha) = 1$ if α is not "short" and we have $c(\alpha_1) = 2$ if α_1 is "short". In this latter case we have $\beta = 2\cdot S_\eta(\alpha_1) = \alpha_0 + S_\eta(\alpha_1)$. These identities imply the claim $\rho_G = \rho_H$.

(c) is a consequence of (a) and the remarks preceding the lemma. □

(3.10) Linear groups. If η is the involution on $G = \mathrm{GL}_n, \mathrm{PGL}_n$ resp. $\mathrm{GL}_n \times \mathbb{G}_m$ as defined above, we can make things more explicit: For $G = \mathrm{GL}_n$ we denote by $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \simeq X^*(T)$ the character $\chi : diag(t_1, \ldots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$. Similarly we denote a character of the group $\mathbb{G}_m^n \times \mathbb{G}_m$ by $(a_1, a_2, \ldots, a_n; a_0) \in \mathbb{Z}^n \times \mathbb{Z}$. The characters of the diagonal torus in PGL_n may be described by the set $\{(a_1, \ldots, a_n) \in \mathbb{Z}^n | \sum_{i=1}^n a_i = 0\}$. The simple roots are of the form $\alpha_i = e_i - e_{i+1} \in X^*(T)$ for the standard basis e_i of $\mathbb{Z}^n$. Since the simple coroots are of the same form $\alpha_i^\vee = e_i - e_{i+1}$, the positive Weyl chamber $X^*(T)^+$ is given by the inequalities $a_1 \geq a_2 \geq \ldots \geq a_n$. The automorphism η acts by $(a_1, a_2, \ldots, a_n) \mapsto (-a_n, -a_{n-1}, \ldots, -a_1)$ (cases GL_n and PGL_n) respectively by $(a_1, \ldots, a_n; a_0) \mapsto (a_0 - a_n, \ldots, a_0 - a_1; a_0)$ in the case $G = \mathrm{GL}_n \times \mathrm{GL}_1$. We have $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right)$ for GL_n and PGL_n resp. $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}; 0\right)$ for $\mathrm{GL}_n \times \mathrm{GL}_1$.

If $\chi + \rho_G = (b_1, \ldots, b_n)$ in the case GL_n and PGL_n we call $S_\chi = \{b_1, \ldots, b_n\} \subset \frac{1}{2}\mathbb{Z}$ the **describing set** of $\chi \in X^*(T)^+$. If $\chi + \rho = (b_1, \ldots, b_n; b_0)$ for $G = \mathrm{GL}_n \times \mathrm{GL}_1$ we call $S_\chi = \left\{b_1 - \frac{b_0}{2}, \ldots, b_n - \frac{b_0}{2}\right\} \subset \frac{1}{2}\mathbb{Z}$ the **describing set** of $\chi \in X^*(T)^+$ and $w(\chi) = b_0$ the **weight** of χ.

Since the entries of $\chi + \rho$ are in a strictly decreasing order it is clear that $\chi \in X^*(T)^+$ is determined uniquely by the set S_χ in the cases GL_n and PGL_n. In the case $\mathrm{GL}_n \times \mathrm{GL}_1$ the character χ is determined uniquely by the pair $(S_\chi, w(\chi))$.

Lemma 3.11. *Let $G = PGL_{2g+1}$ and $H = Sp_{2g}$ be the stable η-endoscopic group. The map $\chi \mapsto S_\chi$ gives a bijection between*

- *the set $X^*(T_H)^+ = X^*(T)^\eta \cap X^*(T)^+$ and*
- *the set of characteristic sets of type C_g.*

Proof: In view of the description of the η-action on $X^*(T)$ given above and of $\rho_G = (g, \dots, 1, 0, -1, \dots, -g)$, it is clear that the describing set S_χ is a characteristic set of type C_g. It is easy to see, that the map is a bijection. $\square$

Lemma 3.12. *Let $G = GL_{2g} \times \mathbb{G}_m$ and $H = GSpin_{2g+1}$ its stable η-endoscopic group. The map $\chi \mapsto (S_\chi, w(\chi))$ gives a bijection between*

- *the set $X^*(T_H)^+ = X^*(T)^\eta \cap X^*(T)^+$ and*
- *the set of pairs (S, w) where $w \in \mathbb{Z}$ and S is a characteristic set of type B_g for w even and S is a characteristic set of type D_g for w odd.*

We omit the easy proof. $\square$

(3.13) Even GSO. Let $T_{GSO} = \{(diag(t_1, \dots, t_{2g}), t_0) \mid t_i t_{2g+1-i} = t_0\}$ be the diagonal torus in $H = \mathrm{GSO}_{2g}$. Using the identification $T_{GSO} = \mathbb{G}_m^{g+1}$ given by the map $(t_0; t_1, \dots, t_g) \mapsto diag(t_1, \dots, t_g, t_0 t_g^{-1}, \dots, t_0 t_1^{-1})$ we get an isomorphism $X^*(T_{GSO}) = \mathbb{Z} \times \mathbb{Z}^g$. Thus $(n_0; n_1, \dots, n_g)$ denotes the character $(diag(t_1, \dots, t_{2g}), t_0) \mapsto \Pi_{i=0}^g t_i^{n_i}$. The $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \le i \le g-1$ and $\alpha_g^\vee = e_{g-1} - (e_0 - e_g)$ are the simple coroots with respect to some splitting of the form $(T_{GSO}, B_H, \{X_\alpha\})$, where B_H denotes the Borel of upper triangular matrices. Thus the positive Weyl chamber $X^*(T_{GSO})^+$ is described by the conditions $n_1 \ge \ldots \ge n_{g-1} \ge n_g \ge n_0 - n_{g-1}$. Let $\sigma \in O_{2g}$ be the block diagonal matrix $(E_{g-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E_{g-1})$. Then the conjugation by σ is an outer automorphism of H which preserves some splitting as above. It acts on $X^*(T_{GSO})$ by $\sigma^* : (n_1, \dots, n_g; n_0) \mapsto (n_1, \dots, n_{g-1}, n_0 - n_g; n_0)$. A character $\chi = (n_0; n_1, \dots, n_g) \in X^*(T_{GSO})^+$ is called σ-positive resp. σ-semipositive if $n_g - \frac{n_0}{2} > 0$ resp. $n_g - \frac{n_0}{2} \ge 0$. It is clear, that each $\chi \in X^*(T_{GSO})^+$ fulfilles exactly one of the following three conditions: $\sigma^*\chi = \chi$, χ is σ-positive, $\sigma^*\chi$ is σ-positive.

(3.14) General Spin groups. Under the identification $X^*(T_{GSO}) = \mathbb{Z}^{g+1} = \oplus_{i=0}^g \mathbb{Z} e_i$ and the dual identification $X_*(T_{GSO}) = \mathbb{Z}^{g+1}$ the system of roots resp. coroots can be written as

$$\begin{aligned} \Phi(GSO) &= \{\pm(e_i - e_j) | 1 \le i < j\} \cup \{\pm(e_i + e_j - e_0) | 1 \le i < j\} \subset X^*(T_{GSO}) \\ \Phi(GSO)^\vee &= \Phi_{SO} = \{\pm e_i \pm e_j | 1 \le i < j \le g\} \subset X_*(T_{GSO}). \end{aligned}$$

If GSpin_{2g} denotes the dual group of GSO_{2g} we get a root datum, where the roles of Φ and $\Phi^\vee$ are interchanged, i.e. $\Phi(\mathrm{GSpin})^\vee = \Phi(GSO)$ and $\Phi(\mathrm{GSpin}) = \Phi_{SO}$. The

projection $\mathrm{Spin}_{2g} \to \mathrm{SO}_{2g}$ induces the following inclusion on the level of cocharacter groups of maximal tori:

$$X_*(T_{Spin}) \;=\; \left\{ \sum_{i=1}^{g} a_i e_i \in \mathbb{Z}^g \;\middle|\; \sum_{i=1}^{g} a_i \equiv 0 \; mod\; 2 \right\} \quad \hookrightarrow \quad X_*(T_{SO}) = \bigoplus_{i=1}^{g} \mathbb{Z} e_i.$$

Then we may define an isogeny $\pi : \mathbb{G}_m \times \mathrm{Spin}_{2g} \to \mathrm{GSpin}_{2g}$ by presribing its effect π_* on the level of cocharacter groups of maximal tori:

$$\begin{array}{rcl} X_*(\mathbb{G}_m) \times X_*(T_{Spin}) & \to & X_*(T_{GSpin}) = \mathbb{Z}^{g+1}, \\ \left(a_0, e_0, \sum_{i=1}^{g} a_i e_i \right) & \mapsto & a_0 e_0 + \sum_{i=1}^{g} a_i e_i - \frac{1}{2} \cdot \left(\sum_{i=1}^{g} a_i \right) \cdot e_0. \end{array}$$

In fact it is easy to check that π_* respects the coroots and its dual π^* respects the roots. From $Hom(\ker(\pi), \mathbb{G}_m) = \mathrm{coker}(\pi^*) \cong \mathbb{Z}/2\mathbb{Z}$ we deduce $\ker(\pi) \cong \mu_2$, and a more careful analysis implies that μ_2 is embedded diagonally in $\mathbb{G}_m \times \mathrm{Spin}_{2g}$. Thus the dual group GSpin_{2g} of GSO_{2g} is in fact what is usually called GSpin_{2g}. With

$$X^*(T_{Spin}) \;=\; \left\{ \sum_{i=1}^{g} a_i e_i \;\middle|\; a_i \in \frac{1}{2}\mathbb{Z}, \quad a_1 \equiv \ldots \equiv a_g \; mod\; \mathbb{Z} \right\} \;\supset\; X^*(T_{SO}) = \bigoplus_{i=1}^{g} \mathbb{Z} e_i$$

the induced map π^* on the level of character groups of maximal tori is given by:

$$\begin{array}{rrcl} \pi^* : & X^*(T_{GSpin}) = \mathbb{Z}^{g+1} & \to & X^*(\mathbb{G}_m) \times X^*(T_{Spin}) = \mathbb{Z} \times X^*(T_{Spin}) \\ & (m_0, m_1, \ldots, m_g) & \mapsto & \left(m_0; m_1 - \frac{m_0}{2}, \ldots, m_g - \frac{m_0}{2} \right). \end{array}$$

The inclusion $i : X^*(T_{GSpin}) = X_*(T_{GSO}) \hookrightarrow X_*(T_{\hat{G}}) = X^*(T_G)$ where T_G is the maximal split torus in $G = \mathrm{GL}_{2g} \times \mathbb{G}_m$, is given by the formula

$$(m_0, m_1, \ldots, m_g) \quad \mapsto \quad (m_1, \ldots, m_g, m_0 - m_g, \ldots, m_0 - m_1; m_0).$$

The half sum of positive roots for $G = \mathrm{GL}_{2g} \times \mathbb{G}_m$ and for GSpin is given by

$$\begin{array}{rcl} \delta_G & = & \left(g - \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, \frac{1}{2} - g; 0 \right) \in X^*(T_G) \otimes \mathbb{Q}, \\ \delta_{GSpin} = & & (0; g-1, \ldots, 1, 0) \in \mathbb{Z}^{g+1} = X^*(T_{GSpin}). \end{array}$$

Note that the composed map:

$$\iota : X^*(T_G)^\eta \otimes \mathbb{Q} \to (X^*(\mathbb{G}_m) \times X^*(T_{Spin})) \otimes \mathbb{Q}, \quad \chi \mapsto \pi^*(i^{-1}(\chi + \delta_G) - \delta_{GSpin})$$

is of the form

$$(m_1,\dots,m_g,m_0-m_g,\dots,m_0-m_1;m_0)\mapsto\left(m_0;m_1-\frac{m_0-1}{2},\dots,m_g-\frac{m_0-1}{2}\right),$$

and thus maps the lattice $X^*(T_G)^\eta$ to the lattice $X^*(\mathbb{G}_m)\times X^*(T_{Spin})$. For $\chi\in X^*(T_G)^\eta$ with odd m_0 we have $\iota(\chi)\in\mathbb{Z}^{g+1}=X^*(\mathbb{G}_m)\times X^*(T_{SO})$, so that $\iota(\chi)$ descends to a character of the maximal torus of $\mathbb{G}_m\times\mathrm{SO}_{2g}$. Let
$X^*(T_H)^{++}=X^*(T_G)^\eta\cap X^*(T_G)^+$ and $X^*(T_H)^{++odd}=\{\chi\in X^*(T_H)^{++}|\ w(\chi)\text{ odd}\}$.
For $\chi\in X^*(T_H)^{++odd}$ let $V_{H,\chi}$ be the finite dimensional irreducible representation of $H=\mathbb{G}_m\times\mathrm{SO}_{2g}$ with highest weight $\iota(\chi)$.

Lemma 3.15. *Let $G=GL_{2g}\times\mathbb{G}_m$ and $H=\mathbb{G}_m\times SO_{2g}$.*
(a) $X^*(T_H)^{++}$ is the set of all σ-semipositive elements in $X^*(T_H)^+$, and $X^*(T_H)^{++odd}$ may also be described as the set of all σ-positive elements in $X^*(T_H)^+$ with odd $w(\chi)$.
(b) The maps $\chi\mapsto V_{H,\chi}$, $\quad\chi\mapsto\tilde V_\chi$, $\quad\chi\mapsto(S_\chi,w(\chi))$ give bijections between

- *the set $X^*(T_H)^{++odd}$ of σ-positive dominant characters with odd $w(\chi)$,*
- *the isomorphism classes of finite dimensional irreducible representations of H with σ-positive highest weight on SO_{2g} and odd weight on $\mathbb{G}_m$,*
- *the isomorphism classes of η-extended finite dimensional irreducible representations of $\tilde G$ which restrict to an odd character of $\mathbb{G}_m$,*
- *the set of pairs (S,w) where S is a characteristic set of type D_g and $w\in\mathbb{Z}$ is odd.*

Proof: (a) $\chi\in X^*(T_H)^+$ means $n_1\geq\ldots\geq n_{g-1}\geq n_g\geq n_0-n_{g-1}$ and χ is σ-semipositive if $n_g\geq n_0-n_g$. Both conditions are satisfied if and only if $\chi\in X^*(T_G)^\eta\cap X^*(T_G)^+$. For odd $w(\chi)$ each σ-semipositive χ is already σ-positive.

(b) may be deduced easily from the definitions and the considerations above. □

4 Real representations

(4.1) Weil group homomorphisms Let $W_\mathbb{R}=\mathbb{C}^*\cup\mathbb{C}^*\sigma$ be the Weil group of $\mathbb{R}$ with the multiplication rules $\sigma\cdot c=\bar c\cdot\sigma$ for $c\in\mathbb{C}^*$ and $\sigma^2=-1\in\mathbb{C}^*$. Recall that the group of continuous homomorphisms from $\mathbb{C}^*$ to $\mathbb{C}^*$ is isomorphic to

$$\Xi\ =\ \left\{(p,q)\in\mathbb{C}^2\ |\quad p-q\in\mathbb{Z}\right\}$$

in such a way that (p,q) corresponds to the homomorphism

$$z\ \mapsto\ z^{(p,q)}:=z^p\cdot\bar z^q:=z^{p-q}\cdot(z\bar z)^q.$$

Similarly the group of continuous homomorphisms from $\mathbb{C}^*$ to a complex torus $\hat{T}$ is isomorphic to

$$X_*(\hat{T}) \otimes_{\mathbb{Z}} \Xi \;=\; \left\{(\mu,\nu) \in (X_*(\hat{T}) \otimes \mathbb{C})^2 \mid \mu - \nu \in X_*(\hat{T})\right\}.$$

For $G = \mathrm{GL}_n \times \mathbb{G}_m$ with diagonal torus T we associate to $\chi \in X^*(T) \otimes \mathbb{Q}$ with $2\chi \in X^*(T)^\eta \cap X^*(T)^+$ the following continuous homomorphism, if $\chi + \rho_G = (b_1, \ldots, b_n; b_0) \in (\frac{1}{2}\mathbb{Z})^{n+1}$:

$$\xi_\chi : \mathbb{C}^* \to \hat{G} = \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^*, \quad z \mapsto diag(z^{b_1}\,\bar{z}^{b_n}, z^{b_2}\,\bar{z}^{b_{n-1}}, \ldots, z^{b_n}\,\bar{z}^{b_1}; |z|^{2b_0}).$$

For a characteristic set S we let $\epsilon_S = 1$ if $S \subset \mathbb{Z}$ (i.e. if S is of type D_g or C_g) and $\epsilon_S = -1$ if $S \subset \frac{1}{2} + Z$ (i.e. if S is of type B_g). We have $b_i + b_{n+1-i} = b_0$ and $b_i = \frac{b_0}{2} + \beta_i$ with $\beta_{n+1-i} = -\beta_i \in S_\chi$. Thus $(-1)^{(b_i, b_{n+1-i})} = (-1)^{b_i - b_{n+1-i}} = (-1)^{2\beta_i} = \epsilon_{S_\chi}$ and $\xi_\chi(-1) = (\epsilon_{S_\chi} \cdot Id_n, 1) \in \mathrm{GL}_n \times \mathbb{G}_m$.

Then ξ_χ may be extended to a homomorphism also called ξ_χ from $W_\mathbb{R}$ to $\hat{G}$ in such a way that $\xi_\chi(\sigma)$ is of the form $(J_\chi, 1) \in \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^*$ where $J_\chi = (a_i \cdot \delta_{i,n+1-j})$ is an antidiagonal matrix, whose entries satisfy

$$a_i \cdot a_{n+1-i} = \epsilon_{S_\chi}.$$

The conjugacy class of ξ_χ does not depend on the special choice of the a_i in the case that n is even. For odd $n = 2g+1$ the conjugacy class only depends on a_{g+1}.

Lemma 4.2. *Let $n = 2g$ in the situation above. There exists a conjugate of ξ_χ which factorizes through $\hat{H} = GSp_{2g}(\mathbb{C})$ if and only if S_χ is of type B_g. There exists a conjugate of ξ_χ which factorizes through $\hat{H} = GSO_{2g}(\mathbb{C})$ if and only if S_χ is of type D_g.*

Proof: With respect to the standard basis $(e_i)_{1 \le i \le 2g}$ the standard symplectic form b_a satisfies $b_a(e_i, e_j) = \varepsilon_i \cdot \delta_{i+j,2g+1}$ with $\epsilon_i = -\epsilon_{2g+1-i} \in \{\pm 1\}$, i.e. $\epsilon_i \cdot \epsilon_{2g+1-i} = -1$ for all i. If S_χ is of type B_g then b_0 is odd and ξ_χ respects this form:

$$\begin{aligned}
b_a(\xi_\chi(z)e_i, \xi_\chi(z)e_j) &= 0 = |z|^{2b_0} \cdot b_a(e_i, e_j) \quad \text{ for } i+j \ne 2g+1 \\
b_a(\xi_\chi(z)e_i, \xi_\chi(z)e_{2g+1-i}) &= b_a(z^{b_i}\bar{z}^{b_{2g+1-i}}e_i, z^{b_{2g+1-i}}\bar{z}^{b_i}e_{2g+1-i}) = |z|^{2b_0} b_a(e_i, e_{2g+1-i}), \\
b_a(\xi_\chi(\sigma)e_i, \xi_\chi(\sigma)e_j) &= 0 = b_a(e_i, e_j) \quad \text{ for } i+j \ne 2g+1 \\
b_a(\xi_\chi(\sigma)e_i, \xi_\chi(\sigma)e_{2g+1-i}) &= b_a(a_{2g+1-i}e_{2g+1-i}, a_i e_i) = b_a(e_i, e_{2g+1-i}),
\end{aligned}$$

since $a_{2g+1-i}a_i = -1$ and $b_a(e_{2g+1-i}, e_i) = -b_a(e_i, e_{2g+1-i})$. Similarly, if S_χ is of type D_g, then ξ_χ respects the standard symmetric form $q_s(e_i, e_j) = \delta_{i+j,2g+1}$.

Conversely, if there exists a conjugate of ξ_χ which factorizes through $\hat{H} = GSp_{2g}(\mathbb{C})$, then ξ_χ respects some non degenerate alternating form q_a'. For $i + j \ne 2g+1$ we

have $z^{b_i}\bar{z}^{b_{2g+1-i}} \cdot z^{b_j}\bar{z}^{b_{2g+1-j}} \neq |z|^{2b_0}$ for some $z \in \mathbb{C}^*$ and therefore the invariance under $\xi_\chi(z)$ implies $q_a'(e_i, e_j) = 0$. Since q_a' is non degenerate we get $q_a'(e_i, e_{2g+1-i}) = -q_a'(e_{2g+1-i}, e_i) \neq 0$. The invariance under $\xi_\chi(\sigma)$ then implies $-1 = a_{2g+1-i}a_i = \epsilon_{S_\chi}$, and S_χ must be of type B_g. Finally if there exists a conjugate of ξ_χ which factorizes through $\hat{H} = GSO_{2g}(\mathbb{C})$, then the same reasoning implies that S_χ is of type D_g. □

(4.3) Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra of $\mathfrak{g}$, which is isomorphic to the W-invariant part of the symmetric algebra $\mathfrak{S}(\mathfrak{t})$ by Harish-Chandra's isomorphism [Hum72, 23.3.]. Here $\mathfrak{t} = Lie(T) \otimes \mathbb{C} = X_*(T) \otimes \mathbb{C}$. Thus every $\chi \in X^*(T) \otimes \mathbb{C}$ gives rise to a homomorphism from $\mathcal{Z}(\mathfrak{g}) = \mathfrak{S}(\mathfrak{t})^W$ to $\mathbb{C}$. Recall that each irreducible $\mathfrak{g}$-module gives rise to a character of $\mathcal{Z}(\mathfrak{g})$.

For simplicity put $\tilde{K}_\infty = K_\infty \cdot Z_\infty$, where $K_\infty \subset G(\mathbb{R})$ is maximal under the connected compact subgroups and where Z_∞ denotes the real connected component of the center of $G(\mathbb{R})$.

Definition 4.4. *Let V_χ be the irreducible algebraic representation of G of highest weight $\chi \in X^*(T)^+$. An irreducible admissible $(\mathfrak{g}, \tilde{K}_\infty)$-module π_∞ is called cohomological with respect to V_χ if and only if there exists some $i \geq 0$ such that $H^i(\mathfrak{g}, \tilde{K}_\infty, V_\chi \otimes \pi_\infty) \neq 0$. An irreducible admissible representation of $G(\mathbb{R})$ is called cohomological with respect to V_χ if and only if its associated $(\mathfrak{g}, \tilde{K}_\infty)$-module of $\tilde{K}_\infty$-finite vectors is cohomological with respect to V_χ. An irreducible automorphic representation $\pi = \hat{\otimes}\pi_v$ of $G(\mathbb{A})$ is called cohomological with respect to V_χ if and only if π_∞ is cohomological with respect to V_χ.*

Lemma 4.5. *Let V_χ be the irreducible algebraic representation of G of highest weight $\chi \in X^*(T)^+$. Let π_∞ be an irreducible admissible $(\mathfrak{g}, K_\infty)$-module which is cohomological with respect to V_χ.*

(a) $\mathcal{Z}(\mathfrak{g})$ acts by the character $\chi + \delta_G$ on V_χ;

(b) $\mathcal{Z}(\mathfrak{g})$ acts by the character $-(\chi + \delta_G)$ on π_∞;

Proof: (a) This is well known ([Hum72, 23.3.]).
(b) This is a consequence of Wigners Lemma ([BoW80, Cor. I.4.2]). □

Lemma 4.6. *For an irreducible admissible representation π_∞ of $G(\mathbb{R})$ let $\phi = \mathcal{L}(\pi_\infty)$ be some associated homomorphism $\phi : W_\mathbb{R} \to \hat{G}$ in the sense of [Lan73].*

(a) If $G = \mathbb{G}_m$ and $\pi_\infty = \sigma_{s,\epsilon}$ with $\sigma_{s,\epsilon}(r) = |r|^s \cdot (sign(r))^\epsilon$ with $s \in \mathbb{C}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ then we have $\phi(z) = (z\bar{z})^s$ and $\phi(\sigma) = (-1)^\epsilon$.

(b) For $G = GL_n$ we have $\mathcal{L}(\omega_{\pi_\infty}) = \det \circ \mathcal{L}(\pi_\infty)$.

(c) For $G = GL_n \times \mathbb{G}_m$ let $\mathcal{L}(\pi_\infty \times \rho_\infty) = \xi_\chi$ in the notations of 4.1. Then we have $\rho_\infty(r) = |r|^{b_0}$ and $\omega_{\pi_\infty}(-1) = (-\epsilon_{S_\chi})^g$ for even $n = 2g$ and $\omega_{\pi_\infty}(-1) = (-1)^g \cdot a_{g+1}$ for odd $n = 2g + 1$.

(d) *If* $\phi(z) = z^\mu \cdot \bar{z}^\nu$ *for* $z \in \mathbb{C}^* \subset W_\mathbb{R}$ *and* $\mu, \nu \in X_*(\hat{T}) \otimes \mathbb{C} = X^*(T) \otimes \mathbb{C}$ *then* π_∞ *has infinitesimal character* μ.

Proof: (a) may be deduced from [Lan73, 2.8.] and (b) from [Lan73, p.30 condition (ii)]. (c) is a consequence of (a) and (b), the explicit description of ξ_χ in 4.1 and the fact that the determinant of the standard antidiagonal matrix of size $2g$ resp. $2g+1$ is $(-1)^g$. (d) is known for discrete series representations for all G: see [Kn86, Theorem 9.20] and observe that the characterization of discrete series representations via characters of maximal anisotropic tori in [Kn86, Theorem 12.7] is the same as in [Lan73]. Then one checks immediately that the statement is invariant under parabolic induction and under forming Langlands quotients. $\square$

(4.7) For an irreducible admissible representation ρ of $\mathrm{GL}_m(F_v)$ resp. a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A})$ and a natural number $n \geq 1$ we denote by $\pi = MW(\rho, n)$ the irreducible admissible representation of $\mathrm{GL}_{mn}(F_v)$ resp. of $\mathrm{GL}_{mn}(\mathbb{A})$, which is the unique irreducible (Langlands) quotient of

$$\pi|\cdot|^{\frac{n-1}{2}} \times \pi|\cdot|^{\frac{n-3}{2}} \times \ldots \times \pi|\cdot|^{\frac{1-n}{2}}.$$

Thus $MW(\rho, 1) = \rho$. Recall the main result of [MW89] that the discrete spectrum $L_{2,disc}(\mathrm{GL}_N(\mathbb{Q})\backslash \mathrm{GL}_N(\mathbb{A}))$ is the Hilbert direct sum of all $MW(\rho, n)$ where n runs over all divisors of N and ρ over all cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ where $N = mn$.

Lemma 4.8. *Let* $m, n \geq 1$ *be natural numbers. Let* $G_m = GL_m \times \mathbb{G}_m$ *and* $\mathfrak{g}_m$ *the Lie algebra of* $G_m(\mathbb{R})$. *Let* $T_m \subset G_m$ *denote the standard maximal split torus.*

(a) *For a cuspidal automorphic representation* $\rho = \hat{\otimes}_v \rho_v$ *of* $GL_m(\mathbb{A})$ *we have* $MW(\rho, n) = \hat{\otimes}_v MW(\rho_v, n)$.

(b) *If* ρ_∞ *is cohomological with respect to* V_χ *for some* $\chi \in X^*(T_m)^\eta \cap X^*(T_m)^+$, *then we have:* $MW(\rho_\infty, n)$ *is cohomological with respect to to* V_σ *for* $\sigma \in X^*(T_{mn})^+$ *if and only if* σ *has the same weight as* χ *and if we have the identity* $S_\sigma = MW(S_\chi, n)$ *between the characteristic sets.*

$\square$

(4.9) More representations. For irreducible representations $\pi_1, \ldots, \pi_k$ of $\mathrm{GL}_{n_i}(F)$ we denote by $\pi_1 \times \ldots \times \pi_k$ the normalized parabolically induced representation of $\mathrm{GL}_n(F)$, where $n = n_1 + \ldots + n_k$.

For $s \in \mathbb{C}$ and an integer $p > 0$ let $D(s,p)$ be the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$, which is the infinite dimensional submodule of $\sigma_{\frac{s+p}{2}, p+1} \times \sigma_{\frac{s-p}{2}, 0}$. Note that the central character of $D(s,p)$ is $\sigma_{s,p+1}$. Viewed as a representation of $\mathrm{SL}_2(\mathbb{R})^\pm = \{A \in \mathrm{GL}_2(\mathbb{R}) | \det(A) = \pm 1\}$ it is independend of s and will be denoted by $D(p)$. As a representation of $\mathrm{SL}_2(\mathbb{R})$ id decomposes $D(p) = D(p)^+ \oplus D(p)^-$, where in $D(p)^+$

the lowest positive $\mathrm{SO}_2(\mathbb{R})$-weight is $p+1$. For $w \in \mathbb{Z}$ we have $\mathcal{L}(D(w,p)\otimes\sigma_{w,0}) = \xi_\chi$ for $\chi + \delta_{\mathrm{GL}_2} = (\frac{w+p}{2}, \frac{w-p}{2}; w)$.

If $n = 2g$ and $\chi + \delta_G = (b_1, \ldots, b_{2g}; b_0)$ is as in 4.1, where $G = \mathrm{GL}_n \times \mathbb{G}_m$, then we may form the tempered representation

$$D_\chi \quad = \quad D(b_0, b_1 - b_{2g}) \times \ldots \times D(b_0, b_g - b_{g+1}) \otimes \sigma_{b_0,0} \qquad \text{of } G(\mathbb{R}).$$

We have $\mathcal{L}(D_\chi) = \xi_\chi$, and D_χ is the only tempered representation of $\mathrm{GL}_{2g}(\mathbb{R})$ which is cohomological with respect to V_χ ([GR11, Thm 4.12.(3)]).

If $n = 2g+1$ and $\chi + \delta_G = (a_g, \ldots, a_1, 0, -a_1, \ldots, -a_g)$, where $G = \mathrm{PGL}_{2g+1}$ then we may form the tempered representation

$$(\mathbf{i}) \qquad D_\chi \;=\; D(S_\chi) \quad := \quad D(0, 2a_g) \times \ldots \times D(0, 2a_1) \times \sigma_{0,g}.$$

As a representation of $\mathrm{GL}_{2g+1}(\mathbb{R})$ it has trivial central character and is thus a representation of $\mathrm{PGL}_{2g+1}(\mathbb{R})$. Again $\mathcal{L}(D_\chi) = \xi_\chi$.

Proposition 4.10. *Let $G = PGL_{2g+1}$ and S be an n-admissible characteristic set of type C_m, where $2g+1 = n\cdot(2m+1)$. Let V_χ be an irreducible representation of G with $S_\chi = MW(n,S)$. Let $\pi_{\infty,1} = D(S)$ be the tempered representation of $PGL_{2m+1}(\mathbb{R})$ as in (*$\mathbf{i}$*) and $\pi_\infty = MW(n, \pi_{\infty,1})$. Then we have:*

$$tr(\eta, H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)) \;=\; \pm 2^g.$$

Proof: Since the kernel μ_{2g+1} of the projection $\mathrm{SL}_{2g+1} \to G = \mathrm{PGL}_{2g+1}$ satisfies $\mu_{2g+1}(\mathbb{R}) = \{1\}$ and $H^1(\mathbb{R}, \mu_{2g+1}) = \{1\}$, we get an isomorphism for the real points $G(\mathbb{R}) \cong \mathrm{SL}_{2g+1}(\mathbb{R})$, and consequently $K_\infty = \mathrm{SO}_{2g+1}$. Furthermore $\mathfrak{g} = \mathfrak{sl}_{2g+1}$. Recall that $\eta(g) = J\cdot\theta(g)\cdot J^{-1}$, where $J = J_{2g+1} \in \mathrm{SO}_{2g+1}$ is anti-diagonal and satisfies ${}^tJ = J = J^{-1}$, and where $\theta : g \mapsto {}^tg^{-1}$ is the Cartan involution. Let $\mathfrak{k} = \mathfrak{so}_{2g+1} = \{A \in \mathfrak{sl}_{2g+1} \mid {}^tA = -A\}$ and $\mathfrak{p} = \{A \in \mathfrak{sl}_{2g+1} \mid {}^tA = A\}$ be the eigenspaces of θ in $\mathfrak{g}$. Assume that a maximal Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ is chosen in such a way, that $Ad(J)$ acts as identity on $\mathfrak{t}$, i.e. $\mathfrak{t} \subset \mathfrak{k}^+ = \{k \in \mathfrak{k} | Ad(J)(k) = k\}$. In fact if we decompose $\mathbb{R}^{2g+1}$ into the ± 1 eigenspaces under the involution J (of dimensions g and $g+1$), then $A \in End(\mathbb{R}^{2g+1})$ is fixed under $Ad(J)$ if it respects this decomposition into eigenspaces. Thus $\mathfrak{k}^+ \cong \mathfrak{so}_g \times \mathfrak{so}_{g+1}$ and we can easily find a maximal torus $\mathfrak{t}$ of rank g inside $\mathfrak{k}^+$.

Then we have $\eta(x) = x$ for each $x \in \mathfrak{t}$. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{sl}_{2g+1}(\mathbb{C})$ be the complexification of $\mathfrak{g}$. For a fixed $x \in \mathfrak{t}$ let $\mathfrak{l} = \{A \in \mathfrak{g} \mid [x, A] = 0\}$, let $\mathfrak{l}_\mathbb{C} = \mathfrak{l} \otimes \mathbb{C}$ and let $\mathfrak{u}_\mathbb{C}$ be the sum of all positive eigenspaces of $ad(ix)$ in $\mathfrak{g}_\mathbb{C}$. Then $\mathfrak{q}_\mathbb{C} = \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u}_\mathbb{C}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$ in the sense of [VZ84]. Now let us assume that $\pi_\infty = A_\mathfrak{q}(\lambda)$ in the notations of [VZ84, Thm. 5.3.], which will be proved later. Recall the isomorphism of [VZ84, Thm. 5.5.]:

$$H^i(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty) \cong H^{i-R}(\mathfrak{l}_\mathbb{C}, \mathfrak{l}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}, \mathbb{C}) \cong Hom_{\mathfrak{l}_\mathbb{C}\cap\mathfrak{k}_\mathbb{C}}(\Lambda^{i-R}(\mathfrak{l}_\mathbb{C} \cap \mathfrak{p}_\mathbb{C}), \mathbb{C}),$$

where $R = \dim(\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}})$, which is given by a cap product with an element of $\Lambda^R(\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}})$ ([VZ84, Proof of Thm 3.3]). Observe that $\mathfrak{k}, \mathfrak{p}, \mathfrak{l}, \mathfrak{u}_{\mathbb{C}}, \mathfrak{q}_{\mathbb{C}}$ are all η-stable, since we have $\eta(x) = x$ and since $\eta = Ad(J) \circ \theta$, where $J \in K_\infty$. If $\epsilon \in \{\pm 1\}$ denotes the eigenvalue of η on $\Lambda^R(\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}})$, we immediately get

$$tr(\eta, H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)) \ = \ \epsilon \cdot tr(\eta, H^*(\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}, \mathbb{C})).$$

Now let $L^K \subset K_\infty$ be the connected Lie subgroup with real Lie algebra $\mathfrak{l} \cap \mathfrak{k}$, and let L^c be a compact Lie group containing L^K with real Lie algebra $i(\mathfrak{l} \cap \mathfrak{p}) \oplus (\mathfrak{l} \cap \mathfrak{k}) \subset \mathfrak{g}_{\mathbb{C}}$. Thus the complexified Lie algebra of L^c is $\mathfrak{l}_{\mathbb{C}}$. We may define these groups as connected components of some stabilizers

$$\begin{aligned} L^K &= \{A \in K_\infty | Ad(A)(x) = x\}^\circ \\ L^c &= \left\{A \in G(\mathbb{C}) | \bar{A} = \theta(A),\ Ad(A)(x) = x\right\}^\circ \end{aligned}$$

We will see later that the stabilizers are already connected in the examples we consider, and then $Ad(J)x = x$ implies $J \in L^K$.
By [GHV73, Theorem V in 4.19.] we have an isomorpism

$$H^i(\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}, \mathbb{C}) \cong H^i(L^c/L^K, \mathbb{C}).$$

There exists involutions η and θ on the Lie group L^c, which induces the involutions with the same name on the Lie algebra, and the isomorphisms above are equivariant with respect to these actions. Since η and the Cartan involution θ differ by conjugation with J, which lies in the path connected group L^K, they are homotopic as automorphisms of L^c and thus induce the same map on cohomology. Since the Cartan involution acts by -1 on $\mathfrak{l} \cap \mathfrak{p}$, the Lefschetz number of η is the sum of the dimensions of the cohomology groups $H^i(L^c/L^K, \mathbb{C})$:

$$tr(\eta, H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)) \ = \ \epsilon \cdot Betti(L^c/L^K),$$

where we denote by $Betti(X) = \sum_i \dim\left(H^i(X, \mathbb{C})\right)$ the sum of all Betti numbers of a topological space X.

Recall $2g + 1 = (2m + 1) \cdot n$. Now we fix $x \in \mathfrak{t}$ in such a way that it is conjugate (under K_∞) to a block diagonal $\tilde{x}$ consisting of blocks

$$\begin{pmatrix} 0_n & x_1 1_n \\ -x_1 1_n & 0_n \end{pmatrix}, \ldots, \begin{pmatrix} 0_n & x_m 1_n \\ -x_m 1_n & 0_n \end{pmatrix}, 0_n$$

with $0 < x_1 < \ldots < x_m$. Then $\mathfrak{l}$ is isomorphic to the algebra $\tilde{\mathfrak{l}}$ of all block diagonal matrices of trace 0 with blocks of the form

(ii) $$\begin{pmatrix} B_1 & C_1 \\ -C_1 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} B_m & C_m \\ -C_m & B_m \end{pmatrix}, B_0,$$

where $B_0, \dots, B_m, C_1, \dots, C_m$ are real $n \times n$ matrices. We have $\mathfrak{l} \cap \mathfrak{k} \cong \tilde{\mathfrak{l}} \cap \mathfrak{k}$ and the latter is characterized by the conditions ${}^tB_j = -B_j$ and ${}^tC_j = C_j$ for all j i.e. $B_0 \in \mathfrak{so}_n$ and $B_\nu + iC_\nu \in \mathfrak{u}_n$ for $\nu = 1, \dots, m$.
Thus we have $L^K = (U_n)^m \times \mathrm{SO}_n$, and the same computations imply that this L^K is the full stabilizer of x in K_∞, so that $J \in L^K$. Similarly we get $L^c = S((U_n \times U_n)^m \times U_n)$, where we write $S(H) = H \cap \mathrm{SL}_{2g+1}$ for a subgroup $H \subset \mathrm{GL}_{2g+1}$. The embedding $L^K \hookrightarrow L^c$ is of the form $U_n \ni A \mapsto (A, \bar{A}) \in U_n \times U_n$ in each factor. Thus $L^c/L^K = S(U_n^m \times (U_n/\mathrm{SO}_n))$. Using the finite covering $U_1 \times S(U_n^m \times (U_n/\mathrm{SO}_n)) \to U_n^m \times (U_n/\mathrm{SO}_n)$ and the Kunneth formula we get $Betti(L^c/L^K) = \frac{1}{2} \cdot Betti(U_n^m \times (U_n/\mathrm{SO}_n))$. Now U_n has the same cohomology as $S^1 \times S^3 \times \ldots \times S^{2n-1}$, ([GHV76, Thm. IX in 6.27]), and since $n = 2\nu + 1$ is odd, the homogeneous space U_n/SO_n has the same cohomology as $S^1 \times S^5 \times S^9 \times \ldots S^{4\nu+1}$ ([Bo53, Prop. 31.4], compare [GHV76, Table I after 11.16]). Thus $Betti(U_n) = 2^n$ and $Betti(U_{2\nu+1}/SO_{2\nu+1}) = 2^{\nu+1}$, and the Kunneth formula finally implies $tr(\eta, H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)) = \epsilon \cdot \frac{1}{2} \cdot (2^n)^m \cdot 2^{\nu+1} = \epsilon \cdot 2^g$, since we have $g = mn + \nu$.

We still have to prove $\pi_\infty = A_{\mathfrak{q}}(\lambda)$ in the sense of [VZ84, Thm. 5.3.]: $\tilde{\mathfrak{l}}$ is the Lie algebra of the group $\tilde{L} = S(\Pi_{j=1}^m \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{R}))$: A block diagonal matrix as in (**ii**) belongs to $\tilde{L}$ if and only if $\prod_{j=1}^m |\det(B_j + C_j)|^2 \cdot \det(B_0) = 1$. Let H be the Cartan subgroup of all such block diagonal matrices where each B_j and each C_j is diagonal and let $\mathfrak{h}$ be its Lie algebra. Thus $H \cong S((\mathbb{C}^*)^{mn} \times (\mathbb{R}^*)^n)$. Write

$$S = \{-a_m, \dots, -a_1, 0, a_1, \dots, a_m\} \text{ with } 0 = a_0 < a_1 < \ldots < a_m.$$

Since S is n-admissible we have $a_j \geq a_{j-1} + n$ for all $j = 1, \dots, m$. Thus the numbers $d_j = a_j - j \cdot n$ satisfy $0 = d_0 \leq d_1 \leq \ldots \leq d_m$. Then we can write $\chi \in X^*(T)^\eta \subset \mathbb{Z}^{2g+1}$ in the form

$$\chi = (d_m, \dots, d_m, \dots, d_1, \dots, d_1, 0, \dots, 0, -d_1, \dots, -d_1, \dots, -d_m, \dots, -d_m),$$

where each entry is repeated n times. With the choice of a Cartan subgroup H as above and a suitable choice of the positive roots for H in $\mathfrak{g}_\mathbb{C}$ it is clear that χ (the highest weight of V_χ) is the restriction of the following unitary character λ of $\tilde{L}$:

$$\lambda : \tilde{L} \ni (X_1, \dots, X_m, B_0) \quad \mapsto \quad \prod_{j=1}^m (\det(X_j)/\overline{\det(X_j)})^{d_j} \qquad \text{for } X_j \in \mathrm{GL}_n(\mathbb{C}).$$

Now $A_{\mathfrak{q}}(\lambda)$ is the representation defined in [VZ84, Thm. 5.3.]. To prove that it equals π_∞ we use the description in [VZ84, Thm. 6.16.]: We decompose $H = T^+ \cdot A^d$, where A^d consists of all block matrices $B = B(b_{k,j})$ as in (**ii**) such that $C_j = 0$ and $B_j = diag(b_{1,j}, \dots, b_{n,j})$ with $b_{k,j} > 0$, and where $T^+ \cong (S^1)^{nm}$ is compact. The unipotent subgroup N^L consists of all block matrices (**ii**), where each B_j is unipotent upper triangular and each C_j is nilpotent upper triangular. Since the restriction of λ to A^d is trivial, the character ν^d of [VZ84] is half the sum of all roots of A^d on $\mathfrak{n}^L$

and this is the character sending the $B = B(b_{k,j})$ to

$$\prod_{j=1}^{m}\left(\prod_{k=1}^{n} b_{k,j}^{2k-n-1}\right)\cdot\prod_{k=1}^{n} b_{0,k}^{k-\frac{n+1}{2}}.$$

The centralizer of A^d is of the form $A^d \cdot M^d$ where $M^d \cong (\mathrm{SL}_2^{\pm}(\mathbb{R}))^{mn} \times \{\pm 1\}^n$. On M^d we take the discrete series representation σ which is the tensor product of the representations $D(2d_j)$ of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ on the factor with indices $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$ and of the representation $sign^g$ on $\{\pm 1\}$.

The group A of [VZ84, (6.8.)] consists of all block matrices B as above, where $B_0 = B_1 = \ldots = B_m$ is a diagonal matrix with positive entries. Its centralizer in GL_{2g+1} is of the form $(\mathrm{GL}_{2m+1})^n$, and therefore it has Langlands decomposition AM with $M \cong (\mathrm{SL}_{2m+1})^n$. The one dimensional representation $\tilde{\nu}^d : (\mathrm{GL}_{2m+1})^n \to \mathbb{R}_{>0}$, $(X_1,\dots,X_n) \mapsto \prod_{k=1}^{n} |\det(X_k)|^{k-\frac{n+1}{2}}$ restricts to the character ν^d on A^d. Now let $N^d \subset G$ be the unipotent radical of some parabolic P^d which has $M^d \cdot A^d$ as Levi factor and let N be the unipotent radical of some parabolic $P \supset P^d$, which has MA as Levi factor. Then $A_{\mathfrak{q}}(\lambda)$ is the Langlands quotient of the representation $Ind_{M^dA^dN^d}^{G}(\sigma \otimes \nu^d \otimes 1)$. But by induction by stages this is the Langlands quotient of $Ind_{MAN}^{G}\tilde{\pi}_\infty$, where $\tilde{\pi}_\infty = (\pi_{\infty,1} \times \ldots \times \pi_{\infty,1}) \otimes \tilde{\nu}^d$ and $\pi_{\infty,1} = D(S)$ is the unitarily induced (tempered) representation as in (**i**). Therefore $A_{\mathfrak{q}}(\lambda) = MW(n, \pi_{\infty,1}) = \pi_\infty$. $\square$

(4.11) Recall that in the case $G = \mathrm{GL}_{2g}$ we have $\tilde{K}_\infty = \mathrm{SO}_{2g} \cdot Z_\infty$ with Z_∞ the connected component of the center of $G(\mathbb{R})$. Put $\tilde{K}_\infty^m = O_{2g} \cdot Z_\infty$. Then $\tilde{K}_\infty^m$ acts on $Hom_{\tilde{K}_\infty}(\Lambda^*(\mathfrak{g}/\tilde{\mathfrak{k}}), V_\chi \otimes \pi_\infty)$ by $(k\phi)(l) = k(\phi(k^{-1}l))$. Since $\tilde{K}_\infty$ acts trivially we get an action of $\tilde{K}_\infty^m/\tilde{K}_\infty = O_{2g}/SO_{2g} = \mathbb{Z}/2\mathbb{Z}$, and we may decompose the cohomology into the $+1$ and -1 eigenspaces $H^*(\mathfrak{g}, \tilde{K}_\infty, V_\chi \otimes \pi_\infty)^\epsilon)$ (here $\epsilon = \pm$) under the action of the non trivial element of $\tilde{K}_\infty^m/\tilde{K}_\infty$. Since $\tilde{K}_\infty^m$ is stable under η and since η acts as identity on $\tilde{K}_\infty^m/\tilde{K}_\infty$, the action of η respects the decomposition of $H^*(\mathfrak{g}, \tilde{K}_\infty, V_\chi \otimes \pi_\infty)$ into these eigenspaces.

Proposition 4.12. *Let $G = GL_{2g}$ and let S be an n-admissible characteristic set of type B_m or D_m, where $g = n \cdot m$. Let V_χ be the irreducible representation of G with describing set $MW(n, S)$ and of weight w. Let $\pi_{\infty,1}$ be a tempered representation of GL_{2m} of type S with central character of weight $-w$, and let $\pi_\infty = MW(n, \pi_{\infty,1})$. Then we have:*

$$tr(\eta, H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)^\epsilon) \;=\; \pm 2^{g-1} \qquad \textit{for } \epsilon = \pm.$$

Proof: At first we note that π_∞ is an induced representation:

$$\pi_\infty = Ind_{G(\mathbb{R})^\circ}^{G(\mathbb{R})}\pi_\infty^\circ, \qquad \text{where } G(\mathbb{R})^\circ = \{g \in \mathrm{GL}_{2g}(\mathbb{R}) \,|\, \det(g) > 0\}.$$

This may be deduced from the analogous statement for the building constituents $D(p_i)$ of $\pi_{\infty,1}$. From this fact one concludes that we have η-equivariant isomorphisms

for both choices of ϵ:

$$H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty)^\epsilon \cong H^*(\mathfrak{g}, K_\infty, V_\chi \otimes \pi_\infty^\circ).$$

Now Z_∞ acts trivially on $\Lambda^*(\mathfrak{g}/\mathfrak{k})$ and by assumption also on $V_\chi \otimes \pi_\infty^\circ$, so that equivariance under Z_∞ is automatically fulfilled. Therefore we may pass to the derived group situation: With $G^{der} = \mathrm{SL}_{2g}$, $\mathfrak{g}^{der} = \mathfrak{sl}_{2g}$ and $K_\infty^{der} = \mathrm{SO}_{2g}$ we get:

$$tr(\eta, H^*(\mathfrak{g}, \tilde{K}_\infty, V_\chi \otimes \pi_\infty)^\epsilon) = tr(\eta, H^*(\mathfrak{g}^{der}, K_\infty^{der}, V_\chi \otimes \pi_\infty^\circ)).$$

Now we can apply the results of [VZ84], and the rest of the proof is analogous to the proof of the preceding proposition with the following modifications: Since J_{2g} is skew symmetric, the Cartan subalgebra $\mathfrak{t}$ can be chosen to be the algebra of all antidiagonal skew symmetric matrices. In the definition of $\tilde{x}$ we have to omit the last 0_n block, and therefore in (**ii**) the last entry A_0 disappears. Thus $L^K \cong (U_n)^m$ and $L^c \cong S((U_n \times U_n)^m)$ and therefore $L^K/L^c \cong S((U_n)^m)$ and

$$tr(\eta, H^*(\mathfrak{g}^{der}, K_\infty^{der}, V_\chi \otimes \pi_\infty^\circ)) = \pm Betti(S((U_n)^m)) = \frac{\pm Betti((U_n)^m)}{2}$$
$$= \pm\frac{(2^n)^m}{2} = \pm 2^{g-1}.$$

□

5 Statement of the Main Theorem

(5.1) For $d \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ let SO_{2g}^d be the quasisplit form of SO_{2g} with splitting field $\mathbb{Q}(\sqrt{d})$. Let G_1 be one of the groups Sp_{2g} (case C_g), SO_{2g+1} (case B_g) or SO_{2g}^d (case D_g). Let $G = \mathrm{PGL}_{2g+1}$ in case C_g and $G = \mathrm{GL}_{2g}$ in the cases of orthogonal groups. Let $\iota : \hat{G}_1 \hookrightarrow \hat{G}$ be the corresponding embedding of dual groups. Let $\chi_d : \mathbb{A}^*/\mathbb{Q}^* \to \{\pm 1\}$ be the quadratic character associated to the field extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ by class field theory.

Definition 5.2. *Let $\pi_1 = \hat{\otimes}_v \pi_{1,v}$ be an irreducible automorphic representation of $G_1(\mathbb{A})$ and $\pi = \hat{\otimes}_v \pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$.*

(a) π is called a weak lift of π_1 if π_v is the lift of $\pi_{1,v}$ for almost all finite places v, where π_v and $\pi_{1,v}$ are unramified.

(b) π is called a semi-weak lift of π_1, if π is a weak lift of π_1 and if

$$\iota \circ \mathcal{L}(\pi_{1,\infty}) = \mathcal{L}(\pi_\infty).$$

(c) (π, π_1) is called an elementary pair for (G, G_1), if π is a semi-weak lift of π_1, if π_1 is cuspidal and generic and if π is cuspidal (and therefore generic) as well.

(d) *Let $S \subset \frac{1}{2}\mathbb{Z}$ be a characteristic set of the same type as (G, G_1). An elementary pair (π, π_1) is called cohomological of type S, if if in case B_g or C_g the representation π_∞ is cohomological with respect to V_χ and if in case D_g for every odd integer w the representation $\pi_\infty \otimes |det|^{-w/2}$ is cohomological with respect to a coefficient system V_χ of weight w with $S = S_\chi$.*

Now we can state our main theorems

Theorem 5.3. *Let $\chi \in X^*(T_H)^+$ be a dominant weight for the group $H = Sp_{2g} \supset T_H$ and let V_χ be the associated highest weight module. Let $\tau = \tau_\infty \otimes \tau_f$ be an irreducible automorphic representation of $Sp_{2g}(\mathbb{A})$ such that τ_f appears with non trivial multiplicity in the alternating sum*

$$\sum_i (-1)^i H^i(\mathfrak{sp}_{2g}, U_g, L_{2,disc}(Sp_{2g}(\mathbb{Q})\backslash Sp_{2g}(\mathbb{A})) \otimes V_\chi)$$

considered as an element in the Grothendieck group. Then there exists a finite family of octuples

$$(X_{\gamma_i}, G^{(i)}, G_1^{(i)}, n_i, d_i, S_i, \pi^{(i)}, \pi_1^{(i)})_{i=1,\ldots,r}$$

such that for each $i = 1, \ldots, r$ we have:
$X_{\gamma_i} \in \{B_{\gamma_i}, C_{\gamma_i}, D_{\gamma_i}\}$ is a type, $(G^{(i)}, G_1^{(i)})$ is an endoscopic pair of type X_{γ_i}, $n_i \in \mathbb{N}_{\geq 1}$, $d_i \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ a square class, S_i is an n_i-admissible characteristic set of type X_{γ_i} and $(\pi^{(i)}, \pi_1^{(i)})$ is an elementary pair for $(G^{(i)}, G_1^{(i)})$, which is cohomological of type S_i, and such that we have:

$$S_\chi \quad = \quad \bigcup_{i=1}^{k} MW(S_i, n_i)$$

is a partition in smaller characteristic sets and such that

$$\tau \text{ lifts weakly to } MW(\tilde{\pi}^{(1)}, n_1) \times \ldots \times MW(\tilde{\pi}^{(r)}, n_r), \quad \text{where}$$

- *If $X_{\gamma_i} = B_{\gamma_i}$, then $\tilde{\pi}^{(i)} = \pi^{(i)}$, $d_i = 1$ and n_i is even.*
- *If $X_{\gamma_i} = C_{\gamma_i}$, then $\tilde{\pi}^{(i)} = \pi^{(i)} \otimes \chi_{d_i}$, and n_i is odd.*
- *If $X_{\gamma_i} = D_{\gamma_i}$, then $\tilde{\pi}^{(i)} = \pi^{(i)}$, $(-1)^{\gamma_i} d_i > 0$, $G_1^{(i)} = SO_{2\gamma_i}^{d_i}$ and n_i is odd.*

We have furthermore: $g = \sum_{i=1}^r \gamma_i \cdot n_i$, $\Pi_{i=1}^r d_i = 1$, $\omega_{\tilde{\pi}^{(i)}} = \chi_{d_i}$.

Theorem 5.4. *Let $\chi \in X^*(T_H)^+$ be a dominant weight for the group $H = Sp_{2g} \supset T_H$ and let V_χ be the associated highest weight module. Assume that there exists a finite family of octuples*

$$(X_{\gamma_i}, G^{(i)}, G_1^{(i)}, n_i, d_i, S_i, \pi^{(i)}, \pi_1^{(i)})_{i=1,\ldots,r}$$

satisfying the conditions stated in Theorem 5.3.
Then there exists an irreducible automorphic representation $\tau = \tau_\infty \otimes \tau_f$ *of* $Sp_{2g}(\mathbb{A})$ *such that* τ_f *appears with non trivial multiplicity in the alternating sum*

$$\sum_i (-1)^i H^i(\mathfrak{sp}_{2g}, U_g, L_{2,disc}(Sp_{2g}(\mathbb{Q})\backslash Sp_{2g}(\mathbb{A})) \otimes V_\chi)$$

and such that

$$\tau \text{ lifts weakly to } MW(\tilde{\pi}^{(1)}, n_1) \times \ldots \times MW(\tilde{\pi}^{(r)}, n_r),$$

where $\tilde{\pi}^{(i)}$ *is related to* $\pi^{(i)}$ *as in theorem 5.3.*

The proofs will be given in (11.6) and (11.7).

6 Comparison of Chevalley volumes

(6.1) Chevalley elements and Chevalley volumes Let $\mathfrak{g}$ be a semisimple split Lie algebra over $\mathbb{R}$ and

$$\mathfrak{g} \quad = \quad \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

a decomposition into a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the corresponding one dimensional root spaces $\mathfrak{g}_\alpha$. We fix some Chevalley basis $Ch_\mathfrak{g} = \{h_\alpha, \alpha \in \Delta\} \cup \{u_\alpha, \alpha \in \Phi\}$ of $\mathfrak{g}$ in the sense of [Hum72, 25.2], where Δ is the set of simple roots with respect to some Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Note that this expecially implies the existence of elements $h_\alpha \in \mathfrak{h}$ for all $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha = \mathbb{R} \cdot u_\alpha$ and

$$[h_\alpha, u_\alpha] = 2u_\alpha; \quad [h_\alpha, u_{-\alpha}] = -2u_{-\alpha}, \quad [u_\alpha, u_{-\alpha}] = h_\alpha$$

and such that the h_α for $\alpha \in \Phi$ lie in the $\mathbb{Z}$-lattice spanned by the h_α for $\alpha \in \Delta$. It is clear that the elements $h_\alpha \in \mathfrak{h}$ and $u_\alpha \wedge u_{-\alpha} \in \Lambda^2\mathfrak{g}$ are uniquely characterized by these conditions. From this we deduce that the Chevalley element

$$\lambda_{Ch} = \bigwedge_{\alpha\in\Delta} h_\alpha \wedge \bigwedge_{\alpha\in\Phi} \quad \in \quad \Lambda^{\dim \mathfrak{g}}\mathfrak{g}$$

(where we do not care about the exact order in the wedge product, so that it is only well defined up to sign) does not depend on the exact choice of the generators $u_\alpha \in \mathfrak{g}_\alpha$. Since all pairs $(\mathfrak{h}, \mathfrak{b})$ are conjugate under inner automorphisms of $\mathfrak{g}$ and since inner automorphisms act as identity on the maximal exterior power $\Lambda^{\dim \mathfrak{g}}\mathfrak{g}$, it is clear that λ_{Ch} is unique up to sign and does not depend on the choices in its construction. It is furthermore clear that to compute λ_{Ch} we just have to check the commutation relations above and do not need to fix the u_α in such a way that the

further conditions in [Hum72, 25.2] are satisfied (commutation relations between u_α and u_β in the case that $\beta \neq \pm\alpha$), since we only need $u_\alpha \wedge u_{-\alpha}$ for our computation.

If $G/\mathbb{R}$ is a semisimple algebraic group with Lie algebra $\mathfrak{g}$ and if $G_K/\mathbb{R}$ is an inner form of G then we have a canonical isomorphism $\Lambda^{\dim G} Lie(G_K) \cong \Lambda^{\dim G}\mathfrak{g}$ (since Ad acts trivially on $\Lambda^{\dim G}\mathfrak{g}$) and thus get a Chevalley element $\lambda_{Ch} \in \Lambda^{\dim G} Lie(G_K)$, which again is unique up to sign. Then we define a Chevalley form to be an G_K-invariant differential form of top degree ω_{Ch} on G_K which is the dual of the Chevalley element λ_{Ch} at the identity of G_K. If G_K is compact then we define the Chevalley volume of G_K (and of its inner form G) to be

$$vol_{Ch}(G_K) \quad := \quad vol_{Ch}(G) \quad := \quad \left| \int_{G_K} \omega_{Ch} \right|.$$

Due to the absolute value this is completely well defined, i.e. does not depend on the sign of the Chevalley element and of the orientation of G_K.

Proposition 6.2. *For all $g \geq 1$ we have*

$$vol_{Ch}(Sp_{2g}) \;=\; vol_{Ch}(Spin_{2g+1}) \;=\; 2 \cdot vol_{Ch}(SO_{2g+1}).$$

Proof: The equation $vol_{Ch}(Spin_{2g+1}) \;=\; 2 \cdot vol_{Ch}(SO_{2g+1})$ is an immediate consequence of the isogeny of degree 2 from $Spin_{2g+1}$ to SO_{2g+1}. The cases $g = 1$ and $g = 2$ are clear, since we have isomorphisms $Spin_3 \cong Sp_2$ and $Spin_5 \cong Sp_4$. We will prove the identity $vol_{Ch}(Sp_{2g}) = 2 \cdot vol_{Ch}(SO_{2g+1})$ by induction on g and first give an outline of the argument:

We will construct standard basis elements for $Lie(SO_{2g+1,K})$ and for $Lie(Sp_{2g,K})$ and denote their wedge product by $\lambda^G_{St} \in \Lambda^{\dim G}\mathfrak{g}$. The invariant differential form ω^G_{St} of top degree which is dual to λ^G_{St} can then be used to define the standard volume $vol_{St}(G_K) = |\int_{G_K} \omega^G_{St}|$. We will prove

(i) $$\omega^{SO_{2g+1}}_{St} \quad = \quad \pm 2^{-(g+1)} \cdot \omega^{SO_{2g+1}}_{Ch} \qquad \text{and}$$

(ii) $$\omega^{Sp_{2g}}_{St} \quad = \quad \pm 2^{-g^2} \cdot \omega^{Sp_{2g}}_{Ch}.$$

The action of SO_n on the unit sphere S^{n-1} with stabilizer SO_{n-1} will imply the relation

(iii) $$vol_{St}(SO_n) \quad = \quad vol(S^{n-1}) \cdot vol_{St}(SO_{n-1})$$

and the action of $Sp_{2g,K}$ on the unit sphere S^{4g-1} in $\mathbb{C}^{2g}$ will imply the relation

(iv) $$vol_{St}(Sp_{2g}) \quad = \quad vol(S^{4g-1}) \cdot vol_{St}(Sp_{2(g-1)}),$$

where we use the Euklidean volume of the spheres, which is known to be

$$vol(S^{n-1}) \quad = \quad \frac{2\cdot\pi^{n/2}}{\Gamma(n/2)}.$$

Thus we get the recursive relations:

$$\begin{aligned} vol_{Ch}(Sp_{2g}) = 2^{g^2}\cdot vol_{St}(Sp_{2g}) &= 2^{2g-1}\cdot 2^{(g-1)^2}\cdot vol(S^{4g-1})\cdot vol_{St}(Sp_{2(g-1)}) \\ &= \frac{2^{2g}\cdot\pi^{2g}}{\Gamma(2g)}\cdot vol_{Ch}(Sp_{2(g-1)}) \end{aligned}$$

and

$$\begin{aligned} vol_{Ch}(SO_{2g+1}) = 2^{-(g+1)}\cdot vol_{St}(SO_{2g+1}) &= \frac{2^{-g}}{2}\cdot vol(S^{2g})\cdot vol(S^{2g-1})\cdot vol_{St}(SO_{2g-1}) \\ &= \frac{2\pi^{g+1/2}\cdot 2\pi^g}{2\cdot\Gamma(g+1/2)\cdot\Gamma(g)}\cdot vol_{Ch}(SO_{2(g-1)+1}). \end{aligned}$$

By the doubling formula $\Gamma(g+1/2)\cdot\Gamma(g) = \frac{\sqrt{\pi}}{2^{2g-1}}\cdot\Gamma(2g)$ the latter simplifies to

$$vol_{Ch}(SO_{2g+1}) \quad = \quad \frac{2^{2g}\cdot\pi^{2g}}{\Gamma(2g)}\cdot vol_{Ch}(SO_{2(g-1)+1})$$

and now the induction step from $g-1$ to g is immediate.

(6.3) Preparation of explicit calculations: For $N\in\mathbb{N}$ we denote by E_{ij} the standard basis elements of the space of $n\times n$ matrices. Let $w_N = (\delta_{i,N+1-j})\in GL_N$ be the standard antidiagonal matrix. For elements $\epsilon_j = \pm 1$ we define $J = (\epsilon_j\delta_{i,N+1-j})$. Then $J^{-1} = {}^tJ = (\epsilon_i\delta_{i,N+1-j})$. The algebraic group $G = \{A\in GL_N|\ {}^tA\cdot J\cdot A = J\}$ has the Lie algebra $\mathfrak{g} = \{A\in Mat_{N\times N}|J^{-1}\cdot{}^tA\cdot J = -A\}$, i.e. a matrix $A = (a_{ij})$ belongs to $\mathfrak{g}$ iff we have

$$\epsilon_i\epsilon_j\cdot a_{N+1-j,N+1-i} = -a_{ij} \quad \text{for all } 1\le i,j\le N.$$

If we define

$$\begin{aligned} u_{ij} &= E_{ij} - \epsilon_i\epsilon_j E_{N+1-j,N+1-i}\in\mathfrak{g} && \text{for } i+j\neq N+1 \\ u_i &= E_{i,N+1-i} && \text{for } 1\le i\le N \end{aligned}$$

then the elements u_{ij} for $i+j<N+1$ and those u_i for which $\epsilon_i\epsilon_{N+1-i} = -1$ form a basis of $\mathfrak{g}$. Introducing the notation $t_i = u_{ii}$ (thus $t_{N+1-i} = -t_i$ and $t_i = 0$ for $2i = N+1$) and $t_{ij} = t_i - t_j$ for $i+j\neq N+1$ we get the commutation relations:

(v) $$[u_{ij}, u_{ji}] = t_{ij}$$

(vi) $$[u_i, u_{N+1-i}] = t_i$$

For $i+j \neq N+1,\ i \neq j$ we have furthermore

$$\textbf{(vii)} \qquad [t_{ij}, u_{ij}] \quad = \quad \begin{cases} 2u_{ij} & \text{if } 2j \neq N+1 \neq 2i \\ u_{ij} & \text{if } 2j = N+1 \text{ or } 2i = N+1. \end{cases}$$

For $2i \neq N+1$ we finally get

$$[t_i, u_i] \quad = \quad 2 \cdot u_i.$$

(6.4) THE CASE Sp_{2g}. Putting $N = 2g$ and $\epsilon_j = -1$ for $j \leq g$ and $\epsilon_j = 1$ for $g+1 \leq j \leq 2g$ we have ${}^tJ = -J$ and get the group $G = Sp_{2g}$. A basis of the Lie algebra $\mathfrak{g}$ consists of the elements u_{ij} for $i+j < 2g+1$ and u_i for $1 \leq i \leq 2g$. If we take $\mathfrak{h}$ to be the subalgebra of diagonal matrices and $\mathfrak{b}$ to be the subalgebra of upper triangular matrices we can use the following basis to compute the Chevalley element (taking into account the commutation relations and $2i \neq 2g+1 \neq 2j$ for all i, j):

$$u_{ij} \text{ for } i \neq j, i+j < 2g+1; \quad u_i \text{ for } 1 \leq i \leq 2g; \quad t_g; \quad t_{i,i+1} \text{ for } 1 \leq i \leq g-1.$$

From $t_{1,2} \wedge \ldots t_{g-1,g} \wedge t_g = u_{11} \wedge \ldots \wedge u_{g-1,g-1} \wedge u_{gg}$ we thus get the Chevalley element

$$\lambda_{Ch} \quad = \quad \pm \bigwedge_{i+j<2g+1} u_{ij} \ \wedge \ \bigwedge_{1 \leq i \leq 2g} u_i.$$

The inner forms of G can be described as

$$G_K(\mathbb{R}) \quad = \quad \left\{ A \in G(\mathbb{C}) |\ \overline{A} = BAB^{-1} \right\},$$

where $B \in G(\mathbb{C})$ has to satisfy the cocycle relation that $\overline{B} \cdot B$ is trivial in the adjoint group, i.e. $\overline{B} \cdot B$ has to be a central matrix in $G(\mathbb{C})$. In our case $G = Sp_{2g}$ we can take $B = J$ (observe $J^2 = -id$) to get the compact inner form G_K. For symplectic matrices satisfying ${}^tA \cdot J \cdot A = J$ the condition $\overline{A} = JAJ^{-1}$ is equivalent to the unitarity condition ${}^tA \cdot \overline{A} = E$ respectively ${}^t\overline{A} \cdot A = E$. Thus

$$\begin{aligned} G_K(\mathbb{C}) \quad &= \quad \left\{ A \in \mathrm{GL}_{2g}(\mathbb{C}) \mid {}^tA \cdot J \cdot A = J, \qquad {}^t\overline{A} \cdot A = E \right\} \\ Lie(G_K) \quad &= \quad \left\{ A \in Mat_{2g}(\mathbb{C}) \mid J^{-1} \cdot {}^tA \cdot J = -A, \qquad {}^t\overline{A} = -A \right\}. \end{aligned}$$

Then we get the following standard basis for the real Lie algebra $Lie(G_K)$:

$$\begin{array}{rl} it_\nu & \text{for } 1 \leq \nu \leq g \\ u_{\nu\mu} - u_{\mu\nu} & \text{for } \nu < \mu,\ \nu+\mu < 2g+1 \\ i(u_{\nu\mu} + u_{\mu\nu}) & \text{for } \nu < \mu,\ \nu+\mu < 2g+1 \\ u_\nu - u_{2g+1-\nu} & \text{for } 1 \leq \nu \leq g \\ i(u_\nu + u_{2g+1-\nu}) & \text{for } 1 \leq \nu \leq g \end{array}$$

From $(u_{\nu\mu} - u_{\mu\nu}) \wedge i(u_{\nu\mu} + u_{\mu\nu}) = 2i \cdot u_{\nu\mu} \wedge u_{\mu\nu}$ and the corresponding formula for the u_ν we immediately get (observing that we have g^2 positive roots):

$$\lambda_{St} = \pm(2i)^{g^2} \cdot i^g \lambda_{Ch} = \pm(-1)^{g(g+1)/2} \cdot 2^{g^2} \lambda_{Ch}.$$

The claim (**ii**) is just the statement for the dual space.

As subgroup of the standard unitary group the group $G_K(\mathbb{R}) = \mathrm{Sp}_{2g,K}(\mathbb{R})$ acts on the unit sphere $S^{4g-1} = \{(a_\mu) \in \mathbb{C}^{2g} | \quad \sum_\mu |a_\mu|^2 = 1\} \subset \mathbb{C}^{2g} = \mathbb{R}^{4g}$ transitively in such a way that the stabilizer of $e_1 = (1, 0, \dots, 0)$ is isomorphic to $Sp_{2(g-1),K}(\mathbb{R})$. By this action the $4g-1$ standard basis vectors of $Lie(Sp_{2g,K})$ with $\nu = 1$ map to the euclidean base of the tangent space of S^{4g-1} at e_1, while the other standard basis vectors descend to the standard basis of $Lie(Sp_{2(g-1),K})$. From these facts the recurrence formula (**iv**) may be deduced easily.

(6.5) THE CASE SO_{2g+1}. Now we put $N = 2g+1$ and $\epsilon_j = 1$ for $1 \le j \le 2g+1$ in 6.3 and get the split orthogonal group $G = SO_{2g+1} = \{A \in \mathrm{GL}_{2g+1}(\mathbb{R}) | {}^tA \cdot w_N \cdot A = w_N\}$ with Lie algebra $\mathfrak{g} = \mathfrak{so}_{2g+1}$. The elements u_{ij} for $i + j < 2g + 2$ form a basis of $\mathfrak{g}$. The commutation relations (**v**) and (**vii**) now imply, that we can get the Chevalley element as the wedge product of the following basis elements:

$$\begin{array}{rl} u_{ij} & \text{for } i \neq j, \quad i+j < 2g+2, \quad j \neq g+1 \neq i \\ u_{i,g+1} \text{ and } 2 \cdot u_{g+1,i} & \text{for } 1 \le i \le g \\ t_{i,i+1} & \text{for } 1 \le i \le g-1 \quad \text{ and finally} \\ 2t_g. & \end{array}$$

From this we deduce immediately:

$$\lambda_{Ch} \quad = \quad \pm 2^{g+1} \cdot \bigwedge_{i+j<2g+2} u_{ij}.$$

Let $SO_N^K = \{A \in \mathrm{SL}_N(\mathbb{R}) | {}^tA \cdot A = id\}$ be the compact form of G. Then the elements $s_{ij} = E_{ij} - E_{ji}$ for $1 \le i < j \le N$ form a basis of its Lie algebra $\mathfrak{so}_N^K = \{A \in Mat_N(\mathbb{R}) | {}^tA = -A\}$. To get SO_{2g+1}^K as an inner twist of the split group SO_{2g+1} we introduce the block matrices

$$C = \begin{pmatrix} id_g & 0 & \frac{1}{2}w_g \\ 0 & 1 & 0 \\ i \cdot w_g & 0 & -\frac{i}{2}id_g \end{pmatrix} \quad \text{with inverse } C^{-1} = \begin{pmatrix} \frac{1}{2}id_g & 0 & -\frac{i}{2}w_g \\ 0 & 1 & 0 \\ w_g & 0 & i \cdot id_g \end{pmatrix}.$$

Then we have ${}^tC \cdot C = J$, and the matrix

$$B \quad = \quad \overline{C}^{-1} \cdot C \quad = \quad \begin{pmatrix} 0 & 0 & \frac{1}{2}w_g \\ 0 & 1 & 0 \\ 2w_g & 0 & 0 \end{pmatrix}$$

satisfies $\overline{B} \cdot B = id_{2g+1}$ and can thus be used to define the inner form $\widetilde{SO}_{2g+1,K} = \{A \in SO_{2g+1}(\mathbb{C}) | \overline{A} = B \cdot A \cdot B^{-1}\}$. (One may replace B by $-B$ if $\det(B) = -1$ to get an element in $SO_{2g+1}(\mathbb{C})$.) With the notation $\phi(A) = C \cdot A \cdot C^{-1}$ we now have the equivalences:

$$A \in SO_{2g+1}(\mathbb{C}) \Leftrightarrow {}^tA \cdot ({}^tC \cdot C) \cdot A = {}^tC \cdot C \Leftrightarrow {}^t(CAC^{-1}) \cdot (CAC^{-1}) = id_{2g+1}$$

and

$$\overline{A} = B \cdot A \cdot B^{-1} \Leftrightarrow \overline{A} = \overline{C}^{-1} \cdot C \cdot A \cdot (\overline{C}^{-1} \cdot C)^{-1} \Leftrightarrow \overline{CAC^{-1}} = CAC^{-1} \Leftrightarrow \phi(A) = \overline{\phi(A)}.$$

Thus we get an isomorphism

$$\phi : \widetilde{SO}_{2g+1,K} \quad \tilde{\to} \quad SO^K_{2g+1}, \qquad A \mapsto CAC^{-1}.$$

On the Lie algebra this isomorphism is given by the same formula. Its effect on the basis elements u_{ij} is given by the following rules where $1 \leq \mu, \nu \leq g$:

$$\begin{aligned}
\phi(u_{\mu\nu}) &= \frac{1}{2} \cdot (s_{\mu\nu} - s_{2g+2-\nu,2g+2-\mu} - i \cdot s_{\mu,2g+2-\nu} - i \cdot s_{\nu,2g+2-\mu}) \\
\phi(u_{\mu,2g+2-\nu}) &= s_{\mu\nu} + s_{2g+2-\nu,2g+2-\mu} + i \cdot s_{\mu,2g+2-\nu} - i \cdot s_{\nu,2g+2-\mu} \qquad \text{for } \mu < \nu \\
\phi(u_{2g+2-\nu,\mu}) &= \frac{1}{4} \cdot (-s_{\mu\nu} - s_{2g+2-\nu,2g+2-\mu} + i \cdot s_{\mu,2g+2-\nu} - i \cdot s_{\nu,2g+2-\mu}) \quad \text{for } \mu < \nu \\
\phi(u_{\mu,g+1}) &= s_{\mu,g+1} - i \cdot s_{g+1,2g+2-\mu} \\
\phi(u_{g+1,\mu}) &= \frac{1}{2} \cdot (-s_{\mu,g+1} - i \cdot s_{g+1,2g+2-\mu}) .
\end{aligned}$$

Here we put $s_{\mu\mu} = 0$. Especially we have for $1 \leq \mu = \nu \leq g$:

$$\phi(t_\mu) = \phi(u_{\mu\mu}) = (-i) \cdot s_{\mu,2g+2-\mu}.$$

In the case $1 \leq \mu < \nu \leq g$ we get by applying the first equation to $u_{\mu\nu}$ and to $u_{\nu\mu}$, using $s_{\nu\mu} = -s_{\mu\nu}$ and combining this with the next two equations:

$$\phi \left(u_{\mu\nu} \wedge u_{\nu\mu} \wedge u_{\mu,2g+2-\nu} \wedge u_{2g+2-\nu,\mu} \right) = s_{\mu\nu} \wedge s_{2g+2-\nu,2g+2-\mu} \wedge s_{\mu,2g+2-\nu} \wedge s_{\nu,2g+2-\mu}.$$

Similarly we get from the last two equations:

$$\phi \left(u_{\mu,g+1} \wedge u_{g+1,\mu} \right) \quad = \quad (-i) \cdot s_{\mu,g+1} \wedge s_{g+1,2g+2-\mu}.$$

Forming the wedge product of the last three families of equations we obtain:

$$\phi \left(\bigwedge_{\mu+\nu<2g+2} u_{\mu\nu} \right) = \pm(-i)^g \cdot \bigwedge_{1\leq i<j\leq 2g+1} s_{ij} \qquad \text{and finally}$$

$$\phi(\lambda_{Ch}) = \pm 2^{g+1} \cdot \bigwedge_{1\leq i<j\leq 2g+1} s_{ij}.$$

If we take the s_{ij} as standard basis for the compact form SO^K_{2g+1}, we get the claim (**i**). One gets the recurrence relation (**iii**) as in the case of the symplectic group. Now the proposition is proved. $\square$

7 The constants α

(7.1) Let $G = \mathrm{PGL}_{2g+1}/\mathbb{Q}$ and $G_1 = \mathrm{SL}_{2g}/\mathbb{Q}$ be its stable η-endoscopic group, where $\eta : g \mapsto J_{2g+1}^{-1} \cdot {}^t g^{-1} J_{2g+1}$. Let Δ be the set of simple roots of G with respect to the Borel $P_\emptyset$ of upper triangular matrices. Recall that the parabolic subgroups containing $P_\emptyset$ are parametrized by the subsets $I \subset \Delta$ and are denoted by P_I. We may identify the space of η-orbits Δ/η with the set of simple roots of G_1 in such a way that an η-stable $I \subset \Delta$ corresponds to a parabolic subgroup $P_{I,1} \subset G_1$ containing the Borel $P_{\emptyset,1}$ of upper triangular matrices. Since G and G_1 are semisimple, we can arrange, that the open compact subgroups $Z_f \subset Z_G(\mathbb{A}_f)$ are small enough in the sense that the group $\zeta = Z_G(\mathbb{Q}) \cap (K_\infty \cdot Z_\infty \times Z_f)$ is trivial in both cases.

The aim of this chapter is to prove:

Proposition 7.2. *Let $I \subset \Delta$ be η-stable. If $\gamma \in P_I(\mathbb{Q})$ and $\gamma_1 \in P_{I,1}(\mathbb{Q})$ are η-matching and contribute to the topological trace formula and if we use the Chevalley measures on the centralizers inside G_1 and 2 times the Chevalley measures on the twisted centralizers inside G, then we have*

$$\alpha_\infty(\gamma, 1) \quad = \quad \alpha_\infty(\gamma_1, 1).$$

(7.3) Recall from [Wes12, Thm 4.9] the definition

$$\alpha_\infty(\gamma_0, h_\infty) \quad = \quad \frac{O_\eta^\infty(I, \gamma_0, h_\infty)}{d^I_{\zeta,\gamma_0}} \cdot (-1)^{\Delta(\gamma_0,\eta)} \cdot \frac{\# H^1(\mathbb{R}, T)}{vol_{db_\infty}\left((\overline{G^I_{\gamma_0,\eta}})'/\zeta\right)}.$$

The factors on the right hand side will be explained in the following. Thereby we will prove their coincidence for γ and γ_1:

(7.4) For the definition of the sign factor $(-1)^{\Delta(\gamma_0,\eta)}$ we refer to [Wes12, Thm. 4.9] and [Wes12, 3.9]. The coincidence of the sign factors has already be proved in [Wes12, 5.26].

The numbers d^I_{ζ,γ_0} are the cardinalities of certain non empty subsets of $H^1(\langle\eta\rangle, \zeta)$ as defined in [Wes12, 2.24]. But since we assume that ζ is the trivial group, they all have to be 1.

(7.5) Recall that $O_\eta^\infty(I, \gamma, h_\infty)$ is the order of the coset space

$$R^I_{\gamma,\eta} \quad = \quad L^m_{\gamma,\eta} \backslash \left(\tilde{L}^m / \tilde{L}\right)^{\eta_\gamma}$$

where

$$\begin{array}{rclcrcl} \tilde{L} & = & p_1 \cdot K_\infty^I Z_\infty A_I \cdot p_1^{-1}, & & \tilde{L}^m & = & p_1 \cdot K_\infty^{I,m} Z_\infty A_I \cdot p_1^{-1}, \\ L_{\gamma,\eta}^m & = & \tilde{L}^m \cap G_{\gamma,\eta}^I(\mathbb{R}), & & \eta_\gamma(x) & = & (g_\eta\gamma)^{-1} \cdot \eta(x) \cdot g_\eta\gamma, \\ K_\infty^I & = & K_\infty \cap P_I(\mathbb{R}), & & K_\infty^{I,m} & = & K_\infty^m \cap P_I(\mathbb{R}). \end{array}$$

But for G the group $K_\infty = SO_{2g+1} \subset G(\mathbb{R}) = \mathrm{PGL}_{2g+1}(\mathbb{R})$ is already maximal compact, as is the subgroup $K_\infty = U_g \subset G_1(\mathbb{R}) = \mathrm{Sp}_{2g}(\mathbb{R})$. Thus we have $K_\infty = K_\infty^m$ in both cases, which implies $\tilde{L} = \tilde{L}^m$ and therefore $O_\eta^\infty(I, \gamma, 1) = 1 = O_\eta^\infty(I, \gamma_1, 1)$.

(7.6) To handle the two remaining factors we have to calculate the (twisted) centralizers:

$$G_{\gamma,\eta}^I(F) \quad = \quad \{A \in P_I(F) | \eta_\gamma(A) = A\} \quad = \quad \left\{A \in P_I(F) | \eta(A)^{-1} \cdot \gamma \cdot A = \gamma\right\}.$$

Here F denotes a field of characteristic 0. For $I = \Delta$, i.e. $P_\Delta = G$, we omit the index: $G_{\gamma,\eta} = G_{\gamma,\eta}^\Delta$. Since $\eta : A \mapsto J^{-1} \cdot {}^tA^{-1} \cdot J$ is formed with a symmetric and involutive antidiagonal matrix $J = J_{2g+1}$, the twisted centralizers may be written in the form:

$$G_{\gamma,\eta}(F) \; = \; \left\{A \in G(F) | \; {}^tA \cdot J\gamma \cdot A \; = \; J\gamma\right\}.$$

Lemma 7.7. *(a) If $\chi_{I,\alpha}(\mathcal{N}(\gamma)) > 1$ for all $\alpha \in \Delta - I$ and if $\mathcal{N}(\gamma)$ is conjugate to an element of $L_\infty^I = K_\infty^I Z_\infty A_I$, then we have*

$$G_{\gamma,\eta} = G_{\gamma,\eta}^I.$$

Proof: Since the twisted centralizer $G_{\gamma,\eta}$ is contained in the usual centralizer $G_{\mathcal{N}(\gamma)}$ of the norm, it suffices to prove that $G_{\mathcal{N}(\gamma)} \subset P_I$. But this is a consequence of the two conditions: Since $\mathcal{N}(\gamma)$ is conjugate to an element of L_∞^I, it is semisimple and may be conjugated inside $P_I(\mathbb{C})$ to a diagonal matrix δ', where in each block of the Levi M_I the diagonal entries have the same absolute value. Then the condition $\chi_{I,\alpha}(\mathcal{N}(\gamma)) > 1$ for all $\alpha \in \Delta - I$ implies, that the absolute values of these entries are different for different blocks of M_I. Thus the centralizer of δ' is contained in M_I, and the centralizer of $\mathcal{N}(\gamma)$ must be contained in P_I. $\square$

Since the two conditions are satisfied for γ contributing to the topological trace formula [Wes12, Theorem 4.9.], we will assume this in the sequel in proving proposition 7.2. Thus we may compute the twisted centralizers in the full group $G = P_\Delta$ resp. $G_1 = P_{1,\Delta}$ and forget the index I.

Lemma 7.8. *The canonical projection $SL_{2g+1} \to PGL_{2g+1}$ induces an isomorphism:*

$$\left\{A \in SL_{2g+1}(F) | \; {}^tA \cdot J\gamma \cdot A \; = \; J\gamma\right\} \quad \tilde{\to} \quad G_{\gamma,\eta}(F).$$

Proof: If $A \in \mathrm{GL}_{2g+1}(F)$ represents an element of $G_{\gamma,\eta}(F)$, then ${}^t\!A \cdot J\gamma \cdot A = \lambda \cdot J\gamma$ for some $\lambda \in F^*$. Since 2 and $2n+1$ are coprime there exists a unique $\mu \in F^*$, such that the two equations $\det(\mu A) = \mu^{2n+1}\det(A) = 1$ and $\mu^2\lambda = 1$ are satisfied i.e. such that $\mu A \in \mathrm{SL}_{2n+1}(F)$ and ${}^t\!(\mu A) \cdot J\gamma \cdot (\mu A) = J\gamma$. □

(7.9) Recall that $\mathcal{N}(\gamma)$ has a conjugate in L^I_∞. But then $\mathcal{N}(\gamma) = \eta(\gamma) \cdot \gamma = J^{-1} \cdot {}^t\gamma^{-1} \cdot J \cdot \gamma = {}^t(J\gamma)^{-1} \cdot J\gamma$ is semisimple, since every element in L^I_∞ is semisimple. Note that $\mathcal{N}(\gamma)$ is well defined as an element of $\mathrm{SL}_{2g+1}(F) \subset \mathrm{GL}_{2g+1}(F)$, since we have $\eta(z\gamma) \cdot z\gamma = \eta(\gamma)\gamma$ for z in the center of GL_{2g+1} and since $\det(\eta(\gamma)) = \det(\gamma)^{-1}$. For a matrix $M \in \mathrm{GL}_n(F)$ put $Aut(M) = \{A \in GL_n(F) |\ {}^t\!A \cdot M \cdot A = M\}$. If $B \in GL_n(F)$ represents $J\gamma \in \mathrm{PGL}_n(F)$, we thus get $G_{\gamma,\eta}(F) \cong Aut(B) \cap \mathrm{SL}_{2g+1}(F)$ with semisimple ${}^t\!B^{-1} \cdot B$. We note that $A \in Aut(M)$ implies $A \in Aut({}^t\!M)$ by transposing the equation, and thus $A \in Aut(M + {}^t\!M)$ and $A \in Aut(M - {}^t\!M)$.

Lemma 7.10. *Let $B \in GL_n(F)$ with semisimple $S = {}^t\!B^{-1} \cdot B$. Assume that S has block diagonal form $S = diag(S_0, 1_m)$ such that $S_0 - 1_{n-m} \in GL_{n-m}(F)$.*

(a) The matrix B is block diagonal: $B = diag(B_0, B_s)$ where $B_s \in GL_m(F)$ is symmetric, where $S_0 = {}^t\!B_0^{-1} \cdot B_0$ and where the skew symmetric part $P_0 = B_0 - {}^t\!B_0$ lies in $GL_{n-m}(F)$.

(b) We have $Aut(B) = Aut(B_0) \times Aut(B_s)$.

(c) We have $S \in Aut(B - {}^t\!B)$ and $S_0 \in Aut(P_0)$.

(d) We have $Aut(B_0) = \{A \in Aut(P_0) | A \cdot S_0 = S_0 \cdot A\}$.

Proof: (a) If one writes B as a block matrix, then the equation ${}^t\!BS = B$ easily implies, that B has to be block diagonal, since $S_0 - 1_{n-m}$ is invertible, that B_s is symmetric and $S_0 = {}^t\!B_0^{-1} \cdot B_0$. But then $P_0 = B_0 \cdot (1_{n-m} - S_0^{-1})$ has to be invertible too.

(b) The claim is a consequence of the fact that each $A \in Aut(B)$ has to be block diagonal: From $Aut(B) \subset Aut(B - {}^t\!B) = Aut(diag(P_0, 0))$ we deduce that A has to be block lower triangular, since P_0 is invertible, and from $Aut(B) \subset Aut(B + {}^t\!B)$ we deduce that A is block diagonal, since B_s is invertible.

(c) ${}^t\!S \cdot (B - {}^t\!B) \cdot S = {}^t\!B \cdot B^{-1} \cdot B \cdot {}^t\!B^{-1} \cdot B - {}^t\!B \cdot B^{-1} \cdot {}^t\!B \cdot {}^t\!B^{-1} \cdot B = B - {}^t\!B$, and the statement for S_0 is a consequence of this identity.

(d) If $A \in Aut(B_0)$ then $A \in Aut(P_0)$ and

$$ {}^t\!B_0^{-1} \cdot B_0 = {}^t({}^t\!AB_0A)^{-1} \cdot ({}^t\!AB_0A) = A^{-1} \cdot {}^t\!B_0^{-1} \cdot {}^t\!A^{-1} \cdot {}^t\!A \cdot B_0 \cdot A = A^{-1} \cdot {}^t\!B_0^{-1} \cdot B_0 \cdot A $$

implies $AS_0 = S_0A$. Conversely $A \in Aut(P_0)$ means

$$ {}^t\!A \cdot B_0(1_{n-m} - S_0^{-1}) \cdot A = B_0(1_{n-m} - S_0^{-1}). $$

If furthermore $A \cdot S_0 = S_0 \cdot A$ then we get ${}^tA \cdot B_0 \cdot A \cdot (1_{n-m} - S_0^{-1}) = B_0(1_{n-m} - S_0^{-1})$, and then the invertibility of $1_{n-m} - S_0^{-1} = S_0^{-1} \cdot (S_0 - 1_{n-m})$ implies $A \in Aut(B_0)$. $\square$

(7.11) For $\beta \in \mathrm{GL}_{2g+1}(F)$ we denote its image in $\mathrm{PGL}_{2g+1}(F)$ by $\overline{\beta}$ and put $\tilde{\gamma} = \eta(\overline{\beta})^{-1} \cdot \gamma \cdot \overline{\beta} \in \mathrm{PGL}_{2g+1}(F)$. Then we have:

$$\begin{array}{rcll} G_{\tilde{\gamma},\eta} & = & \beta^{-1} \cdot G_{\gamma,\eta} \cdot \beta & \\ \mathcal{N}(\tilde{\gamma}) & = & \beta^{-1} \cdot \mathcal{N}(\gamma) \cdot \beta & \text{([Wes12, 2.3 (14)])} \\ \tilde{B} & := & {}^t\beta \cdot B \cdot \beta \quad \text{represents } J\tilde{\gamma}, & \\ \tilde{S} & = & \beta^{-1} \cdot S \cdot \beta \quad \text{where } \tilde{S} := {}^t\tilde{B}^{-1} \cdot \tilde{B}. & \end{array}$$

By varying γ in its η-conjugacy class we may thus assume, that the semisimple matrix S has block diagonal form $S = diag(S_0, 1_m)$ with invertible $S_0 - 1_{2g+1-m}$, that $B = diag(B_0, B_s)$ is block diagonal with symmetric B_s and that the antisymmetric part $P_0 = B_0 - {}^tB_0$ is the standard antidiagonal symplectic matrix $J_{2p} \in GL_{2p}(F)$ as in 6.4. Then we have $m = 2(g-p)+1$. Now we put $\delta = diag(1_{g-p}, S_0, 1_{g-p}) \in \mathrm{Sp}_{2g} =: G_1$.

Lemma 7.12. *With the notations introduced above we have:*

(a) $G_{\gamma,\eta} \quad \cong \quad SO(B_s) \times \{A \in Sp_{2p} |\ AS_0 = S_0A\}$.

(b) $(G_1)_\delta \quad \cong \quad Sp_{2(g-p)} \times \{A \in Sp_{2p} |\ AS_0 = S_0A\}$.

(c) The elements $\gamma\eta$ and δ are matching.

Proof: (a) Since $Aut(B_0) \subset \mathrm{Sp}_{2p} \subset \mathrm{SL}_{2p}$ lemma 7.10(b) implies

$$G_{\gamma,\eta} \quad = \quad Aut(B) \cap SL_{2g+1} \cong (Aut(B_s) \cap \mathrm{SL}_{2(g-p)+1}) \times Aut(B_0)$$

and the claim is a consequence of lemma 7.10(d).

(b) This is clear, since S_0 does not have the eigenvalue 1.

(c) Since this is essentially [BWW02, prop 6.3.(c)] we sketch the argument: Since the claim may be checked over an algebraic closed field $\bar{F}$, one can assume γ to be a diagonal matrix. Then B is antidiagonal and S is diagonal, and an explicit calculation using the definitions of stable endoscopy implies the claim. $\square$

(7.13) Now we can finish the proof of proposition 7.2: Since the elements in $P_{I,1}(\mathbb{Q})$ matching with $\eta\gamma$ are all stable conjugate we may assume $\gamma_1 = \delta$ in the notations of the preceding lemma.

Recall that in the factor $\# H^1(\mathbb{R}, T)$ the torus T can be chosen to be a maximal torus in the centralizer $G_{\gamma_0,\eta}$. Now we may chose some maximal torus T_0 inside the group $C = \{A \in \mathrm{Sp}_{2p} |\ AS_0 = S_0A\}$, an $\mathbb{R}$-anisotropic maximal torus T_1' in $Sp_{2(g-p)}$ (of rank $g-p$) and an

R-anisotropic maximal torus T' in $SO(B_s)$ again of rank $g-p$. Then we can chose the maximal torus T in $G_{\gamma,\eta}$ in such a way that it maps to $T' \times T_0$ under the isomorphism of 7.12(a) and the maximal torus T_1 in $(G_1)\delta$ in such a way that it maps to $T_1' \times T_0$ under 7.12(b). Since we have $\#H^1(\mathbb{R}, T') = 2^{g-p} = \#H^1(\mathbb{R}, T_1')$ we get immediately $\#H^1(\mathbb{R}, T) = \#H^1(\mathbb{R}, T_1)$.

(7.14) For the remaining factor $vol_{db_\infty}\left((\overline{G^I_{\gamma_0,\eta}})'/\zeta\right)$ we first recall the fact that ζ is the trivial group in both cases. For $G = G^I_{\gamma_0,\eta} = G_{\gamma_0,\eta}$ the group $\overline{G}/\mathbb{R}$ is the inner form of G which is compact modulo the center of G ([Wes12, 3.9]), and then $\overline{G}'$ denotes the common kernel of all rational characters of G (which may be viewed as characters of $\overline{G}$ using the isomorphism $G/G^{der} = \overline{G}/\overline{G}^{der}$). But lemma 7.12 implies that we have $\overline{G_{\gamma,\eta}}' \cong SO_{2(g-p)+1} \times \overline{C}'$ and $\overline{(G_1)_\delta}' \cong Sp_{2(g-p),K}(\mathbb{R}) \times \overline{C}'$, where $Sp_{2(g-p),K}$ is the compact inner form of $Sp_{2(g-p)}/\mathbb{R}$. From prop 6.2 and our assumptions on the measures we now get

$$vol_{db_\infty}\left((\overline{G^I_{\gamma,\eta}})'/\zeta\right) = vol_{db_{\infty,1}}\left((\overline{(G_1)^I_{\gamma_1}})'/\zeta\right).$$

This finishes the proof of proposition 7.2. $\square$

8 Restatement of the Topological Trace formula

(8.1) Now we assume that we are in one of the following situations:

$$\begin{aligned}(G, \eta, G_1) &= (\mathrm{PGL}_{2n+1}, \eta, \mathrm{Sp}_{2n})\\ (G, \eta, G_1) &= (\mathrm{GL}_{2n} \times \mathrm{GL}_1, \eta, \mathrm{GSpin}_{2n+1}) \text{ where } n = 1 \text{ or } n = 2.\end{aligned}$$

We regret that the condition $n = 1$ or $n = 2$ is missing in [Wes12, Theorem 5.23] due to an formulation error: In fact for $n \geq 3$ the group GSpin_{2n+1} is not cohomological trivial (so the stabilization of the trace formula will be more complicated), and the groups K_∞ have not been specified, unless we can use the exceptional isomorphisms $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$ and $\mathrm{GSpin}_3 \cong \mathrm{GL}_2$. Now we may restate [Wes12, Theorem 5.23] and [Wes12, Corollary 5.27]:

Theorem 8.2. *(a) Assume that (G, η, G_1) is as in 8.1. If the G-module M matches with the G_1-module M_1, then*

$$H^*_c\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty, \mathcal{M}\right) \in \mathcal{G}ro(G(\mathbb{A}_f) \rtimes \eta)$$

is the lift of

$$H^*_c\left(G_1(\mathbb{Q})\backslash G_1(\mathbb{A})/K_{\infty,1} Z_\infty, \mathcal{M}_1\right) \in \mathcal{G}ro(G_1(\mathbb{A}_f));$$

(b) The same statement holds if one the cohomology with compact support H_c^* is replaced by the usual cohomology H^*.

In the situation $(G, \eta, G_1) = (\mathrm{PGL}_{2n+1}, \eta, \mathrm{Sp}_{2n})$ we remark that due to proposition 7.2 the notion of lifting depends on a notion of matching test functions on the group of finite adeles which is now defined intrinsically using the Haar measures on the centralizers which are defined using a Chevalley basis.

9 Spectral decomposition of cohomology

(9.1) Frankes Theorem.
The functor $V \mapsto H^*(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty, \mathcal{V})$ induces a homomorphism between Grothendieck groups: $H^{*,G} : \mathcal{G}ro(G, alg) \to \mathcal{G}ro(G(\mathbb{A}_f))$, where $\mathcal{G}ro(G(\mathbb{A}_f))$ denotes the Grothendieck group of admissible representations of $G(\mathbb{A}_f)$ in a complex vector space.
For $I \subset \Delta$ the naive (not normalized) parabolic induction functor induces a homomorphism:

$$Ind_{P_I(\mathbb{A}_f)}^{G(\mathbb{A}_f)} : \quad \mathcal{G}ro(M_I(\mathbb{A}_f)) \quad \longrightarrow \quad \mathcal{G}ro(G(\mathbb{A}_f)).$$

Let $\check{\mathfrak{a}}_I = X^*(P_I) \otimes \mathbb{R} \cong X^*(A_I) \otimes \mathbb{R}$ and $\pi_I : X^*(A_0) \otimes \mathbb{R} \to \check{\mathfrak{a}}_I$ be the canonical projection, which annihilates the simple roots in I. Let $\rho_I = \pi_I(\rho)$, where $\rho = \rho_G \in X^*(A_0) \otimes \mathbb{R}$ is half the sum of the positive roots. Let $\overline{\check{\mathfrak{a}}_I^+}$ be the set of those $\lambda \in X^*(P_I) \otimes \mathbb{R} = \check{\mathfrak{a}}_I$ for which $\langle \lambda, \hat{\alpha} \rangle \geq 0$ for all simple roots $\alpha \in \Delta - I$. Here $\hat{\alpha} = \alpha^\vee$ denotes the corresponding simple coroot.

After tensoring the Grothendieck groups with the field $\mathbb{Q}$ we can define the following homomorphisms for $I \subset \Delta$:

$$\begin{array}{rcl} J_{Eis,I} : \mathcal{G}ro(G, alg)_{\mathbb{Q}} & \to & \mathcal{G}ro(M_I, alg)_{\mathbb{Q}} \\ V & \mapsto & \displaystyle\sum_{\lambda \in \overline{\check{\mathfrak{a}}_I^+}} \frac{1}{n_I(\lambda)} V_{I,\lambda}^* \end{array}$$

where $n_I(\lambda)$ is the number of Weyl chambers to which λ belongs and where $V_{I,\lambda}^*$ is the part of the virtual M_I module $H^*(\mathfrak{n}_I, V)$, on which $\mathfrak{a}_I = X_*(A_I) \otimes \mathbb{R}$ acts by $-\lambda$.
If G is a split group, let W be its Weyl group. In view of Kostants theorem $J_{Eis,I}$ may be rewritten in the form

$$\begin{array}{rcl} J_{Eis,I} : \mathcal{G}ro(G, alg)_{\mathbb{Q}} & \to & \mathcal{G}ro(M_I, alg)_{\mathbb{Q}} \\ V_\chi & \mapsto & \displaystyle\sum_{\substack{w \in W^I, \\ \pi_I(w(\chi+\rho)) \in -\overline{\check{\mathfrak{a}}_I^+}}} \frac{(-1)^{l(w)}}{n_I(w(\chi+\rho)-\rho)} V_{w(\chi+\rho)-\rho}^I \end{array}$$

where V_χ is an irreducible algebraic G module with highest weight χ, where $V^I_{w(\chi+\rho)-\rho}$ denotes the corresponding highest weight module for M_I and where

$$W^I = \{w \in W | w^{-1}\alpha > 0 \text{ for } \alpha \in I\} = \{w \in W | w^{-1}\hat{\alpha} > 0 \text{ for } \alpha \in I\}.$$

If $\chi : Z_\infty A_I \to \mathbb{C}^*$ denotes a continuous character, we define $\mathcal{C}^\infty(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi)$ to be

$$\{f \in \mathcal{C}^\infty(M_I(\mathbb{Q})\backslash M_I(\mathbb{A})) \mid f(zg) = \chi(z)f(g) \text{ for } z \in Z_\infty \cdot A_I\},$$

and denote by $L_2(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi)$ its completion as a Hilbert space with respect to integration over $M_I(\mathbb{Q})\backslash M_I(\mathbb{A})'$, where $M_I(\mathbb{A})'$ is the common kernel of all homomorphisms of the form $m \mapsto |\phi(m)|$, where $\phi : M_I \to \mathbb{G}_m$ is a rational character. By $L_{2,disc}(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi)$ we denote the intersection of $\mathcal{C}^\infty(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi)$ with the discrete spectrum in $L_2(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi)$. Now we can define a homomorphism $H^*_{2,disc} = H^{*,M_I}_{2,disc} : \mathcal{G}ro(M_I, alg) \to \mathcal{G}ro(M_I(\mathbb{A}_f))$, if we put for an irreducible $M_I(\mathbb{Q})$-modul V, on which $Z_\infty \cdot A_I$ acts by the central character χ_V:

$$H^{*,M_I}_{2,disc}(V) \quad = \quad H^*(\mathfrak{m}_I, K_\infty \cap M_I(\mathbb{R}); L_{2,disc}(M_I(\mathbb{Q})\backslash M_I(\mathbb{A}),\chi_V^{-1}) \otimes V).$$

Theorem 9.2. *The following identity between homomorphisms from $\mathcal{G}ro(G, alg)$ to $\mathcal{G}ro(G(\mathbb{A}_f))_\mathbb{Q}$ holds:*

$$H^{*,G} \quad = \quad \sum_{I\subset\Delta} Ind^{G(\mathbb{A}_f)}_{P_I(\mathbb{A}_f)} \circ H^{*,M_I}_{2,disc} \circ J_{Eis,I}.$$

This is a consequence of the spectral sequence [Fr98, (7.4.1)], since the E_1 term coincides with the E_∞ term in the Grothendieck group. The statement is closely related to the considerations in [Fr98, 7.7.(1) and (2)]. $\square$

Lemma 9.3. *For $w \in W^I$ and $\pi_I(w(\chi+\rho)) \in -\overline{\check{\mathfrak{a}}^+_I}$ we have $I = \{\alpha \in \Delta |\, w^{-1}\alpha > 0\}$.*

Proof: For $\alpha \in \Delta$ and $w \in W^I$ write $w^{-1}\hat{\alpha} = \sum_{\beta\in\Delta} k_\beta\hat{\beta}$. Then we have

$$\langle w(\chi+\rho),\hat{\alpha}\rangle \;=\; \langle \chi+\rho, w^{-1}\hat{\alpha}\rangle \;=\; \sum_{\beta\in\Delta} k_\beta\langle\chi+\rho,\hat{\beta}\rangle.$$

Now $\langle\chi+\rho,\hat{\beta}\rangle = \langle\chi,\hat{\beta}\rangle + 1 \geq 1$ since χ is a dominant weight. In the case $w^{-1}\hat{\alpha} > 0$ (especially for $\alpha \in I$) all the k_β are non negative with at least one of them being positive. Consequently $\langle w(\chi+\rho),\hat{\alpha}\rangle > 0$. For $w^{-1}\hat{\alpha} < 0$ we get $\langle w(\chi+\rho),\hat{\alpha}\rangle < 0$ by the analogous argument.

Now write $\pi_I(w(\chi+\rho)) = w(\chi+\rho) - \sum_{\beta\in I} n_\beta\beta$. The condition $\langle\pi_I(w(\chi+\rho)),\hat{\alpha}\rangle = 0$ for $\alpha \in I$ implies that $n_\beta = \sum_{\gamma\in I} c_{\beta\gamma}\langle w(\chi+\rho),\hat{\gamma}\rangle$, where $C = (c_{\beta\gamma})_{\beta,\gamma\in I}$ is the inverse of the matrix $(\langle\beta,\hat{\alpha}\rangle)_{\alpha,\beta\in I}$. It is known that $c_{\beta\gamma} \geq 0$ for all $\beta,\gamma \in I$. For $\alpha \in \Delta - I$ we now deduce from $\pi_I(w(\chi+\rho)) \;\in\; -\overline{\check{\mathfrak{a}}^+_I}$

$$\begin{aligned} 0 \quad &\geq\; \langle\pi_I(w(\chi+\rho)),\hat{\alpha}\rangle \;=\; \langle w(\chi+\rho),\hat{\alpha}\rangle - \sum_{\beta,\gamma\in I}\langle\beta,\hat{\alpha}\rangle\cdot c_{\beta\gamma}\cdot\langle w(\chi+\rho),\hat{\gamma}\rangle \\ &\geq\; \langle w(\chi+\rho),\hat{\alpha}\rangle \end{aligned}$$

since $\langle \beta, \hat{\alpha} \rangle \leq 0$ for $\alpha \notin I, \beta \in I$ and since $c_{\beta\gamma} \geq 0$ and $\langle w(\chi+\rho), \hat{\gamma} \rangle > 0$ for all $\beta, \gamma \in I$. By the computation at the beginning we conclude $w^{-1}\hat{\alpha} < 0$, which immediately implies the claim, since we have $w^{-1}\hat{\alpha} > 0$ for $\alpha \in I$ by the definition of W^I. $\square$

This implies that for each $w \in W$ there is at most one $I \subset \Delta$ such that the condition $w \in W^I$ and the cone condition $\pi_I(w(\chi+\rho)) \in -\overline{\check{\mathfrak{a}}_I^+}$ are both satisfied. We write $I = I(w)$ if both conditions are satisfied.

(9.4) The case of linear groups. In the case of the automorphism η of order 2 on $G = \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_1, \mathrm{PGL}_n$ we can make things more explicit: Recall that for $G = \mathrm{GL}_n$ we denote by $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \simeq X^*(T)$ the character $\chi : diag(t_1, \ldots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$. Similarly we denote a character of the group $\mathbb{G}_m^n \times \mathbb{G}_m$ by $(a_1, a_2, \ldots, a_n; a_0) \in \mathbb{Z}^n \times \mathbb{Z}$. The characters of the diagonal torus in PGL_n may be described by the set $\{(a_1, \ldots, a_n) \in \mathbb{Z}^n | \sum_{i=1}^n a_i = 0\}$. The simple roots are of the form $\alpha_i = e_i - e_{i+1} \in X^*(T)$ for the standard basis e_i of $\mathbb{Z}^n$. The positive Weyl chamber is given by the inequalities $a_1 \geq a_2 \geq \ldots \geq a_n$. The automorphism η acts by $(a_1, a_2, \ldots, a_n) \mapsto (a_n, a_{n-1}, \ldots, a_1)$ (cases GL_n and PGL_n) respectively by $(a_1, \ldots, a_n; a_0) \mapsto (a_0 - a_n, \ldots, a_0 - a_1; a_0)$ in the case $G = \mathrm{GL}_n \times \mathrm{GL}_1$. We have $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right)$. This implies for the Weyl-group action of $w \in W = S_n$:

$$w(\chi+\rho) - \rho \quad = \quad (b_1, \ldots, b_n) \quad \text{with } b_i = a_{w^{-1}(i)} + i - w^{-1}(i)$$

If the complement of $I \subset \Delta$ is written in the form $\Delta - I = \left\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r-1}}\right\}$ with $i_1 < i_2 < \ldots < i_{r-1}$, we introduce the notations $i_0 = 0, i_r = n, \;\; j_\mu = i_\mu - i_{\mu-1}$ and $\tilde{I} = (i_1|i_2 - i_1|\ldots|n - i_{r-1}) = (j_1|\ldots|j_r)$. Then the Levi group M_I, which consists of block diagonal matrices, is isomorphic to $\mathrm{GL}_{j_1} \times \ldots \times GL_{j_r} \subset \mathrm{GL}_n$ (with obvious modifications for the other linear groups under consideration).The parabolic P_I consists of upper triangular block matrices.
If we consider $\chi = (a_1, a_2, \ldots, a_n)$ as a character on $T \subset M_I$ we write it in the form $(a_1, \ldots, a_{i_1}|a_{i_1+1}, \ldots, a_{i_2}|\ldots|a_{i_{r-1}+1}, \ldots, a_n)$ and call the sequence of numbers between two $|$ a block. For χ in the positive Weyl chamber the condition $w \in W^I$ then means that the numbers in each block of $w(\chi+\rho) - \rho = (b_1, \ldots, b_n)$ with $b_i = a_{w^{-1}(i)} + i - w^{-1}(i)$ are in a semi decreasing order: $b_{i_\mu+1} \geq \ldots \geq b_{i_{\mu+1}}$. The numbers in the blocks of $w(\chi+\rho) = (\tilde{b}_1, \ldots, \tilde{b}_n)$ are then in a strictly decreasing order. We have $\tilde{b}_i = \tilde{a}_{w^{-1}(i)}$, where $\tilde{a}_i = a_i + \frac{n+1}{2} - i$ is a strictly decreasing sequence. With the arithmetic means of the numbers in each block

$$m_\mu = \frac{1}{j_\mu} \cdot \sum_{\nu=1}^{j_\mu} \tilde{b}_{i_{\mu-1}+\nu}$$

we can now describe

$$\pi_I(w(\chi+\rho)) = (m_1, \ldots, m_1|m_2, \ldots, m_2|\ldots|m_r \ldots, m_r)$$

The condition $\pi_I(w(\chi+\rho)) \in -\overline{\check{\mathfrak{a}}_I^+}$ is equivalent to the statement that the sequence of rational numbers m_μ is semi increasing: $m_1 \le m_2 \le \ldots \le m_r$.

The $\tilde{b}_i$ forming a strictly decreasing sequence in each block we get the relation $\tilde{b}_{i_\mu} \le m_\mu \le m_{\mu+1} \le \tilde{b}_{i_\mu+1}$ which implies the description $I = \{\alpha_i \in \Delta | \tilde{b}_i > \tilde{b}_{i+1}\} = \{\alpha_i \in \Delta | w^{-1}(i) < w^{-1}(i+1)\}$ as in the preceeding lemma.

10 Partitions

(10.1) NUMBERED AND UNORDERED PARTITIONS. By a numbered partition $J = (J_1, \ldots, J_r)$ of a finite set T we mean an r-tuple of pairwise disjoint subsets J_i such that $T = \cup_{i=1}^r J_i$. The underlying unordered partition is the set $\{J_1, \ldots, J_r\}$. Two numbered partitions $J = (J_1, \ldots, J_r)$ and $J' = (J'_1, \ldots, J'_r)$ are called equivalent (in symbols $J \sim J'$) if $\{J_1, \ldots, J_r\} = \{J'_1, \ldots, J'_r\}$.

(10.2) TOTALLY ORDERED SETS. For a finite totally ordered set T (e.g. a finite subset of $\mathbb{Q}$) let $\eta = \eta_T$ be the unique order reversing involution.
If T and K are totally ordered sets of the same cardinality, then there exists a unique order preserving bijection ι between T and K, and then $\iota \circ \eta_T = \eta_K \circ \iota$. Then ι induces a bijection between the numbered resp. unordered partitions of T and those of K.

Definition 10.3. *Let $J = (J_1, ..., J_r)$ be a numbered partition of the finite totally ordered set T.*

(a) J is called chronological numbered if $i < j$ implies $t_i < t_j$ for all $t_i \in J_i$ and $t_j \in J_j$.

(b) $\eta(J) = (\eta(J_r), \ldots, \eta(J_1))$.

(c) J is called η-stable, if $\eta(J) \sim J$

(d) J is called η-fixed, if $\eta(J) = J$ as a numbered partition.

(e) J is called η-invariant, if $\eta(J_\mu) = J_\mu$ for $\mu = 1, \ldots, r$.

(f) J is called η-admissible, if there is $l \le \frac{r}{2}$ such that $\eta(J_\mu) = J_{r+1-\mu}$ for $\mu = 1, \ldots, l$ and $\eta(J_\mu) = J_\mu$ for $\mu = l+1, \ldots, r-l$.

(10.4) THE PERMUTATION ASSOCIATED TO A PARTITION. It is clear, that for each numbered partition $J = (J_1, \ldots, J_r)$ of a finite ordered set T there exists a unique chronological numbered partition $J' = (J'_1, \ldots, J'_r)$ such that $\#J_\mu = \#J'_\mu$ for $\mu = 1, \ldots, r$. For example if $T = \{1, \ldots, n\}$ we can put $i_\mu = \#J_1 + \ldots + \#J_\mu$ for $\mu = 0, \ldots, r$ and then we have $J'_\mu = \{i_{\mu-1}+1, \ldots, i_\mu\}$. Now we define $w_J \in S_T$ to

be the unique permutation of T such that w_J induces an order preserving bijection from J_μ to J'_μ for all $\mu = 1, \ldots, r$.

(10.5) The partition associated to a permutation. Now we can continue the considerations of (9.4): Let $I \subset \Delta$ be given. Writing $\Delta - I = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r-1}}\}$ with $i_1 < i_2 < \ldots < i_{r-1}$ and $i_0 = 0, i_r = n$ we can consider the chronological numbered partition $J'_\mu = \{i_{\mu-1}+1, \ldots, i_\mu\}$ of $T = \{1, \ldots, n\}$. Let $\chi = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a fixed dominant weight and let $w \in W^I$ be such that $\pi_I(w(\chi+\rho)) \in -\overline{\check{\mathfrak{a}}_I^+}$. We define

$$J_\mu \quad = \quad \{j \in \{1, \ldots, n\} \mid \quad i_{\mu-1} < w(j) \leq i_\mu\} \quad = \quad w^{-1}(J'_\mu)$$

and get a numbered partition $J = (J_1, \ldots, J_r)$. Since $\tilde{a}_i = a_i + \frac{n+1}{2} - i$ is a strictly decreasing sequence and since $\tilde{b}_i = \tilde{a}_{w^{-1}(i)}$ is a strictly decreasing sequence if i varies in some J'_μ, we get that w induces an order preserving bijection from J_μ to J'_μ for each $\mu = 1, \ldots, r$. Consequently w coincides with the element w_J associated to the partition J. Therefore I and w are uniquely determined by J.

We may rewrite the arithmetic means in the following form

$$m_\mu = m(J_\mu) = \frac{1}{\# J'_\mu} \sum_{j \in J'_\mu} \tilde{b}_j = \frac{1}{\# J_\mu} \sum_{j \in J_\mu} \tilde{a}_j.$$

Thus we get that the numbered partition J is admissible numbered in the following sense:

Definition 10.6. *A numbered partition $J = (J_1, \ldots, J_r)$ of $\{1, \ldots, n\}$ is called admissible numbered (with respect to a dominant $\chi = (a_1, \ldots, a_n)$) if the arithmetic means*

$$m(J_\mu) = \frac{1}{\# J_\mu} \sum_{j \in J_\mu} \left(a_j + \frac{n+1}{2} - j \right)$$

form a semi increasing sequence of rational numbers:

$$m(J_1) \leq m(J_2) \leq \ldots \leq m(J_r).$$

(10.7) Reformulation of Frankes theorem. It is clear that each numbered partition is equivalent to some admissible numbered partition. Now let $\mathcal{P}$ be a set of admissible numbered partitions of $\{1, 2, \ldots, n\}$ which contains for every unordered partition exactly one admissible numbered representative. To an admissible numbered partition J we can associate the element $w = w_J \in S_n$ and using the numbers i_μ defining the chronological ordered partition J' we get the subset $I(J) = \Delta - \{\alpha_{i_1}, \ldots, \alpha_{i_{r-1}}\}$ of Δ associated to J.

Proposition 10.8. *For $G = GL_n$ or $G = PGL_n$ or $G = GL_n \times GL_1$ we have the following identity in $\mathcal{G}ro(G(\mathbb{A}_f))$:*

$$H^{*,G}(V_\chi) \quad = \quad \sum_{J\in\mathcal{P}} (-1)^{l(w_J)}\ Ind_{P_{I(J)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{2,disc}^{*,M_{I(J)}}(V_{w_J(\chi+\rho)-\rho}^{I(J)}).$$

Proof: The preceding combinatorial considerations show that the sum over I and $w \in W^I$ in theorem 9.2 may be replaced by a sum over all admissible numbered partitions. But the numbers $n_{I(J)}(w_J(\chi+\rho)-\rho)$ count the admissible numberings of the unordered partition underlying J. Since the usual intertwining operator induces an isomorphism

$$\iota_{J,\tilde J}: \quad Ind_{P_{I(J)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{2,disc}^{*,M_{I(J)}}(V_{w_J(\chi+\rho)-\rho}^{I(J)}) \quad \simeq \quad Ind_{P_{I(\tilde J)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{2,disc}^{*,M_{I(\tilde J)}}(V_{w_{\tilde J}(\chi+\rho)-\rho}^{I(\tilde J)}),$$

if J and $\tilde J$ are two different admissible numberings of the same unordered partition, we get the claim. □

Lemma 10.9. *Let $\chi \in X^*(T)^\eta$ be η-invariant, and let J be an admissible numbered partition of $\{1,\dots,n\}$.*

(a) $m(\eta(J_\mu)) = a_0 - m(J_\mu)$ where $a_0 = 0$ in the cases $G = GL_n$ and $G = PGL_n$.

(b) $\eta(J)$ is an admissible numbered partition.

(c) If J is η-admissible, then it is η-stable.

(d) Each η-stable admissible numbered partition is equivalent to an η-admissible numbered partition.

(e) An η-stable J is η-fixed if and only if it is η-admissible and there is at most one index μ with $\eta(J_\mu) = J_\mu$.

Proof: (a) $\chi = (a_1,\dots,a_n;a_0)$ resp. $\chi = (a_1,\dots,a_n)$ being η-invariant means $a_{\eta(j)} = a_0 - a_j$ for $j = 1,\dots,n$. This implies

$$\tilde a_{\eta(j)} = a_{\eta(j)} + \frac{n+1}{2} - \eta(j) = a_0 - a_j + \frac{n+1}{2} - (n+1-j) = a_0 - \tilde a_j$$

$$m(\eta(J_\mu)) = \frac{1}{\#\eta(J_\mu)} \sum_{j\in\eta(J_\mu)} \tilde a_j = \frac{1}{\#J_\mu} \sum_{j\in J_\mu} \tilde a_{\eta(j)} = a_0 - \frac{1}{\#J_\mu} \sum_{j\in J_\mu} \tilde a_j = a_0 - m(J_\mu).$$

(b) is an immediate consequence of (a).

(c) is clear from the definitions since η is an involution on the set $\{1,\dots,n\}$.

(d) If J is η-stable let l be the number of η-orbits $\{J_\mu, \eta(J_\mu)\}$ containing two elements inside $\{J_1,\dots,J_r\}$. From each of these orbits we pick an element J_μ with $m(J_\mu) \le m(\eta(J_\mu))$ and we renumber these elements by the integers $1,\dots,l$ in such a way

that we have $m(J_1) \le \ldots \le m(J_l)$. We put $J_{n+1-\mu} = \eta(J_\mu)$ for $\mu = 1, \ldots, l$ and we let $J_{l+1}, \ldots, J_{n-l}$ be any numbering of the subsets satisfying $\eta(J_\mu) = J_\mu$. Since (a) implies $m(J_\mu) = \frac{a_0}{2}$ for $\mu = l+1, \ldots, n-l$ and furthermore $m(J_l) \le \frac{a_0}{2} \le m(J_{n+1-l})$ we conclude that we get an admissible numbering, which is η-admissible by construction.

(e) is an immediate consequence of the definitions. □

(10.10) We associate to every η-admissible partition

$$J = (J_1, \ldots, J_l, J_{l+1}, \ldots, J_{r-l}, J_{r-l+1}, \ldots, J_r)$$

an η-fixed partition $\pi(J) = (\tilde{J}_1, \ldots, \tilde{J}_\lambda)$. If $2l = r$ we put $\pi(J) = J$ and $\lambda = 2l = r$. In the case $2l < r$ we put $\tilde{J}_\mu = J_\mu$ for $\mu = 1, \ldots, l$, $\tilde{J}_{l+1} = \bigcup_{\mu=l+1}^{r-l} J_\mu$ and finally $\tilde{J}_\mu = J_{\mu+r-2l-1}$ for $\mu = l+2, \ldots, 2l+1 =: \lambda$.

If $\pi(J)$ is replaced by an equivalent η-fixed partition $\tilde{J}'$ then it is easy to see that there exists an η-admissible partition J' equivalent to J such that $\pi(J') = \tilde{J}'$.

By eventually replacing an admissible partition by an equivalent one we may assume that the set of partitions $\mathcal{P}$ is formed in such a way, that the subset $\mathcal{P}^{stable}$ of η-stable partitions in $\mathcal{P}$ consists of η-admissible partitions and that π maps $\mathcal{P}^{stable}$ to the subset $\mathcal{P}^{fix}$ of η-fixed partitions in $\mathcal{P}$. The subset of η-invariant partitions in $\mathcal{P}$ is denoted by $\mathcal{P}^{inv}$. Observe that $\mathcal{P}^{inv} = \left\{ J \in \mathcal{P}^{stable} | \pi(J) = J_{trivial} \right\}$ where $J_{trivial}$ is the trivial partition $(\{1, \ldots, n\})$.

If J is an η-admissible partition as above with $2l < r$, then $w_{\pi(J)}$ is an order preserving bijection from $\tilde{J}_{l+1}$ to $\{i_l + 1, \ldots, n - i_l\}$. Then the shift $\sigma : i \mapsto i - i_l$ is an order preserving bijection of this latter set to $\{1, \ldots, \tilde{n}\}$ where $\tilde{n} = n - 2i_l$. Now $K_\nu = \sigma(w_{\pi(J)}(J_{l+\nu}))$ for $\nu = 1, \ldots, r-2l$ defines a numbered partition of $\{1, \ldots, \tilde{n}\}$. We write $K(J) = (K_1, \ldots, K_{r-2l})$. It is easy to see that $K(J)$ is admissible numbered with respect to the character $\tilde{\chi}$ which is obtained by subtracting the ρ of $\mathrm{GL}_{\tilde{n}}$ from the middle block of the character $w_{\pi(J)}(\chi + \rho)$.

Lemma 10.11. *Let $J = (J_1, \ldots, J_r)$ be an admissible numbered partition.*

(a) *We have $I(\eta(J)) = \eta(I(J))$ and $\eta(w_J) = w_{\eta(J)}$.*

(b) *If J is η-fixed then we have $w(J) \in W^\eta$ and $I(J)$ is η-invariant.*

(c) *$\eta(K_\nu) = K_\nu$ for $\nu = 1, \ldots, r-2l$.*

(d) *If J is η-admissible then $w(J) \cdot w(\pi(J))^{-1}$ is the identity on $\{1, \ldots, i_l\}$ and on $\{n - i_l + 1, \ldots, n\}$, and we have*

$$\sigma \circ w(J) \cdot w(\pi(J))^{-1} \circ \sigma^{-1} \quad = \quad w_{K(J)}.$$

Proof: (a) To $\eta(J) = (\tilde{J}_1, \ldots, \tilde{J}_r)$ with $\tilde{J}_\mu = \eta(J_{r+1-\mu})$ one associates the numbers $\tilde{i}_\mu = \#\tilde{J}_1 + \ldots + \#\tilde{J}_\mu = \#J_r + \ldots + \#J_{r+1-\mu} = n - (\#J_1 + \ldots + \#J_{r-\mu}) = n - i_{r-\mu}$. Then we have $\eta(I(J)) = \eta\left(\Delta - \left\{\alpha_{i_1}, \ldots, \alpha_{i_{r-1}}\right\}\right) = \Delta - \left\{\alpha_{n-i_1}, \ldots, \alpha_{n-i_{r-1}}\right\} = \Delta - \left\{\alpha_{\tilde{i}_{r-1}}, \ldots, \alpha_{\tilde{i}_1}\right\} = I(\eta(J))$ since $\eta(\alpha_i) = \alpha_{n-i}$. With the involution $w_n : i \mapsto n+1-i$ of $\{1, \ldots, n\}$ we have $\eta(w) = w_n \circ w \circ w_n$ in $W = S_n$. Now w_n maps $\tilde{J}_\mu$ bijectively and order reversing to $J_{r+1-\mu}$. Then w_J maps by definition $J_{r+1-\mu}$ bijectively and order preserving to $\{i_{r-\mu} + 1, \ldots, i_{r-\mu+1}\}$ and finally w_n maps this set bijectively and order reversing to $\{n + 1 - i_{r-\mu+1}, \ldots, n + 1 - (i_{r-\mu} + 1)\} = \left\{\tilde{i}_{\mu-1} + 1, \ldots, \tilde{i}_\mu\right\}$. Thus $\eta(w_J)$ has the defining properties of $w_{\eta(J)}$ and thus must be equal to this permutation.

(b) is a consequence of (a).

(c) Since $w_{\pi(J)}$ and $J_{l+\nu}$ are fixed by $\eta = \eta_n$ we have $\eta_n(w_{\pi(J)}(J_{l+\nu})) = w_{\pi(J)}(J_{l+\nu})$ and from this one concludes $\eta_{\tilde{n}}(K_\nu) = K_\nu$.

(d) again is a consequence of the defining properties of $w_j, w_{\pi(J)}$ and $w_{K(J)}$. □

Lemma 10.12. *If $\tilde{J}$ is an η-fixed partition, then the map $J \mapsto K(J)$ defines a bijection from the inverse image of $\tilde{J}$ under π in $\mathcal{P}^{stable}$ to the set $\mathcal{P}^{inv}_{\tilde{n}}$ of η-invariant partitions of $\{1, \ldots, \tilde{n}\}$.*

□

(10.13) Now let $G = \mathrm{PGL}_{2g+1}$ and $G_1 = \mathrm{Sp}_{2g}$ be its stable η-twisted endoscopic group. We have $X^*(T_1) = X^*(T)^\eta$ and $W(G_1) = W(G)^\eta$. The set of simple roots Δ_1 of G_1 may be identified with the set of η-orbits in Δ. Therefore the subsets I_1 of Δ_1 (which parametrize the standard parabolics of G) may be identified with the η-stable subsets I of Δ. Denote by $P_{1,I}$ resp. $M_{1,I}$ the corresponding parabolic resp. Levi subgroups of G_1.

Proposition 10.14. *For $G_1 = Sp_{2g}$ we have the following identity in $\mathcal{G}ro(G_1(\mathbb{A}_f))$, if $\chi \in X^*(T_1) = X^*(T)^\eta$ is dominant:*

$$H^{*,G_1}(V_\chi) \quad = \quad \sum_{J \in \mathcal{P}^{fix}} Ind^{G_1(\mathbb{A}_f)}_{P_{1,I(J)}(\mathbb{A}_f)} H^{*,M_{1,I(J)}}_{2,disc}(V^{I(J)}_{w_J(\chi+\rho)-\rho}).$$

Proof: This again is a consequence of theorem 9.2, if we take $W(G_1) = W(G)^\eta$ and lemma 10.11 into account. □

Theorem 10.15. *For dominant $\chi \in X^*(T_1) = X^*(T)^\eta$ we have*

$$H^{*,G_1}_{2,disc}(V_\chi) \qquad \textit{lifts to} \qquad \sum_{J \in \mathcal{P}^{inv}} Ind^{G(\mathbb{A}_f)}_{P_{I(J)}(\mathbb{A}_f)} H^{*,M_{I(J)}}_{2,disc}(V_{w_J(\chi+\rho)-\rho})$$

The proof is by induction on g, the case $g = 0$ being trivial. So we assume that the theorem is valid for all $\gamma < g$. Now [Wes12, Cor 5.27] as restated in 8.2(b) implies, that $H^{*,G_1}(V_\chi)$ lifts to $H^{*,G}(V_\chi)$. Write

$$\begin{aligned} R_J &= (-1)^{l(w_J)}\ Ind_{P_{I(J)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{2,disc}^{*,M_{I(J)}}(V_{w_J(\chi+\rho)-\rho}^{I(J)}) \qquad \text{and} \\ R_J^1 &= (-1)^{l_1(w_J)}\ Ind_{P_{1,I(J)}(\mathbb{A}_f)}^{G_1(\mathbb{A}_f)} H_{2,disc}^{*,M_{1,I(J)}}(V_{w_J(\chi+\rho)-\rho}^{I(J)}) \end{aligned}$$

By 10.14 and 10.8 we may reformulate:

$$\sum_{J\in\mathcal{P}^{fix}} R_J^1 \quad \text{lifts to} \quad \sum_{J\in\mathcal{P}} R_J.$$

Here we have to observe that the filtration which induces Frankes spectral sequence is η-invariant. Thus we get an η-action on the initial term of the spectral sequence and therefore on $\sum_{J\in\mathcal{P}} R_J$ in such a way that it is compatible with the η-action on the limit.

Since R_J appears together with $R_{\eta(J)}$ in the sum for G, if J is not η-stable (thus $R_J + R_{\eta(J)}$ has trivial η-traces), we may replace the sum on the right hand side by

$$\sum_{J\in\mathcal{P}^{stable}} R_J = \sum_{J\in\mathcal{P}^{fix}} \sum_{J'\in\mathcal{P}^{stable},\pi(J')=J} R_{J'}.$$

We will prove that the induction assumption implies that for $J \in \mathcal{P}^{fix}$, $J \neq J_{trivial}$ we have: R_J^1 lifts to $\sum_{J'\in\mathcal{P}^{stable},\pi(J')=J} R_J$. Then it is an immediate consequence that $H_{2,disc}^{*,G_1}(V_\chi) = R_{J_{trivial}}^1$ lifts to $\sum_{J\in\mathcal{P}^{inv}} R_J$.

Indeed let $J = (J_1, \ldots, J_r)$ be η-fixed and put $j_i = \#J_i$. Then $j_i = j_{r+1-i}$ implies that $r = 2l + 1$ must be odd, since $\sum_{i=1}^r j_i = 2g + 1$ is odd. Write $j_{l+1} = 2\gamma + 1$. Then $M_{1,I(J)}$ is of the form $L_J \times \mathrm{Sp}_{2\gamma}$ with $L_J = \Pi_{i=1}^l \mathrm{GL}_{j_i} \subset Sp_{2g}$. Similarly the inverse image of $M_{I(J)}$ in GL_{2g+1} is of the form $\widetilde{M_{I(J)}} = L_J \times \mathrm{GL}_{2\gamma+1} \times \eta(L_J)$. For $J' \in \mathcal{P}^{stable}$ with $\pi(J') = J$ we get $\widetilde{M_{I(J')}} = L_J \times M_{K(J')} \times \eta(L_J)$ in the notations introduced above. In the case $J \neq J_{trivial}$ we have $\gamma < g$ and $L_J \neq 1$ and we can apply the induction assumption to the lift from $\mathrm{Sp}_{2\gamma}$ to $\mathrm{PGL}_{2\gamma+1}$ with respect to the character $\tilde{\chi}$ introduced earlier. Observe that the $J' \in \mathcal{P}^{stable}$ with $\pi(J') = J$ are in one-to-one correspondence with the equivalence classes of η-invariant partitions K of $\{1, \ldots, , 2\gamma + 1\}$.

It is an easy exercise that the induction assumption implies that $H_{2,disc}^{*,M_{1,I(J)}}(V_{w_J(\chi+\rho)-\rho}^{I(J)})$ lifts to $\sum_{\pi(J')=J}(-1)^{l(w_{J'})-l_1(w_J)} H_{2,disc}^{*,M_{I(J')}}(V_{w_{J'}(\chi+\rho)-\rho}^{I(J')})$, and then the claim is clear, since the lifting relation remains valid after parabolic induction.

$\square$

11 Proof of the Main Theorem

(11.1) Characters on the blocks. Let $\chi = (a_g, \ldots, a_1, 0, -a_1, \ldots, -a_g) \in X^*(T)^\eta \subset \mathbb{Z}^{2g+1}$ be a dominant weight. Recall that

$$S_\chi = \{-a_g - g, \ldots, -a_1 - 1, 0, a_1 + 1, \ldots, a_g + g\}$$

is the characteristic set of type C_g associated to χ. Let $J = (J_1, \ldots, J_r)$ be an η-invariant partition of $\{1, \ldots, 2g+1\}$. Via 10.2 this gives rise to an η-invariant partition $S_\chi = \Sigma_1 \cup \ldots \cup \Sigma_r$. We remark that every ordering of an η-invariant partition is admissible, but we have to fix one. For $i = 1, \ldots, r$ let $j_i = \#(\Sigma_i)$ and $k_i = -j_1 - \ldots - j_{i-1} + j_{i+1} + \ldots + j_r$. Let $T_i \subset \mathrm{GL}_{j_i}$ be the diagonal torus and let $\kappa_i : \mathbb{G}_m \to \mathbb{G}_m, \quad z \mapsto z^{k_i}$. Observe $k_i \equiv \sum_{\nu \neq i} j_\nu = 2g + 1 - j_i \equiv 1 + \#(\Sigma_i) \mod 2$.

Observe that $M_I(\mathbb{A}) \ni (m_1, \ldots, m_r) \mapsto \prod_{i=1}^r |det(m_i)|^{k_i}$ is the modulus of the action of $M_I(\mathbb{A})$ on the unipotent radical $U_I(\mathbb{A})$ of $P_I(\mathbb{A})$ and thus its square root is the additional factor, which describes the normalized parabolic induction in terms of the naive induction. This square root is attached to the difference $\rho_G - \rho_{M_I}$.

There is exactly one index i such that $0 \in \Sigma_i$. Then $j_i = 2\gamma_i + 1$ is odd and Σ_i is of type C_{γ_i}. We have $\Sigma_i = S_{\tilde{\chi}_i}$ for some η-invariant dominant character $\tilde{\chi}_i$. Since k_i is even in this case, we may put $\chi_i = \tilde{\chi}_i \cdot (\det)^{-k_i/2}$ and get an η-invariant dominant character $\chi_i \times \kappa_i^{-1}$ on $T_i \times \mathbb{G}_m$. If $\Sigma_i = \{-b_{\gamma_i} - \gamma_i, \ldots, -b_1 - 1, 0, b_1 + 1, \ldots, b_{\gamma_i} + \gamma_i\}$ with $b_{\gamma_i} \geq \ldots \geq b_1 \geq 0$ then

$$\chi_i \;=\; \left(b_{\gamma_i} - \frac{k_i}{2}, \ldots, b_1 - \frac{k_i}{2}, -\frac{k_i}{2}, -b_1 - \frac{k_i}{2}, \ldots, -b_{\gamma_i} - \frac{k_i}{2}\right) \in \mathbb{Z}^{2\gamma_i+1}.$$

If $0 \notin \Sigma_i$ then $j_i = 2\gamma_i$ is even, Σ_i is of type D_{γ_i} and k_i is odd. By lemma 3.15(b) there exists an η- invariant dominant character of the form $\chi_i \times \kappa_i^{-1}$ on $T_i \times \mathbb{G}_m$ such that its characteristic set is Σ_i. If $\Sigma_i = \{-b_{\gamma_i} - \gamma_i, \ldots, -b_1 - 1, b_1 + 1, \ldots, b_{\gamma_i} + \gamma_i\}$ with $b_{\gamma_i} \geq \ldots \geq b_1 \geq 0$ then

$$\chi_i \;=\; \left(b_{\gamma_i} + \frac{-k_i + 1}{2}, \ldots, b_1 + \frac{-k_i + 1}{2}, -b_1 + \frac{-k_i - 1}{2}, \ldots, -b_{\gamma_i} + \frac{-k_i - 1}{2}\right) \in \mathbb{Z}^{2g_i}.$$

In both cases we have $\eta(\chi_i) = \chi_i + (k_i, \ldots, k_i) = \chi_i \cdot \kappa_i \circ \det$.

(11.2) Now we want to relate the discrete spectrum cohomology of the Levi M_I to the cohomology of its building blocks GL_{j_i}: Observe that $K_\infty \cap M_I(\mathbb{R})$ is not connected: We have $O_{2g+1} \cap M_I(\mathbb{R}) = \prod_{i=1}^r O_{j_i}$ and thus we may pick orthogonal matrices with determinant -1 at an even number of places to get an element of $K_\infty^I = \mathrm{SO}_{2g+1} \cap M_I(\mathbb{R})$. Therefore we have to take eigenspaces for the action of O_{j_i}/SO_{j_i} on the cohomology of the building blocks as in 4.11. If j_i is odd, then O_{j_i}/SO_{j_i} is represented by central matrices, i.e. the central character of a representation decides whether an eigenspace is empty or not.

Proposition 11.3. *In the notations of theorem 10.15 and of the preceding sections we have*

$$H^{*,M_{I(J)}}_{2,disc}(V^{I(J)}_{w_J(\chi+\rho)-\rho}) \quad = \quad \sum_{\epsilon=\pm}\bigotimes_{i=1}^{r} H^{*,GL_{j_i}}_{2,disc}(V_{\chi_i})^{\epsilon}.$$

Proof: It is a matter of book keeping, that we may write:

$$w_J(\chi+\rho)-\rho \quad = \quad (\chi_1|\ldots|\chi_r).$$

If one modifies the left hand side by replacing in $H^*(\mathfrak{m}_I, K^I_\infty A_I \mathbb{Z}_\infty, \ldots)$ the group K^I_∞ by its connected component $(K^I_\infty)^\circ = \prod_i SO_{j_i}$, then the Künneth formula implies, that this is

$$\bigotimes_{i=1}^{r} H^{*,\mathrm{GL}_{j_i}}_{2,disc}(V_{\chi_i}) \quad = \quad \bigotimes_{i=1}^{r}\sum_{\epsilon=\pm} H^{*,\mathrm{GL}_{j_i}}_{2,disc}(V_{\chi_i})^{\epsilon}.$$

Now invariance under $\mathrm{SO}_{2g+1}\cap M_I(\mathbb{R})$ means that only those tensor products of eigenspaces appear, which share the sign. $\square$

Lemma 11.4. *Let $\chi = (a_1,\ldots,a_{2g}) \in X^*(T) = \mathbb{Z}^{2g}$ be a dominant character of the split torus T in GL_{2g} satisfying $a_{2g+1-i}+a_i = w$ for some $w\in\mathbb{Z}$ and let V_χ be the associated algebraic representation with highest weight χ. Then the center of GL_{2g} acts by the character $z\mapsto z^{gw}$.*

Proof: This is clear since we have $\sum_{i=1}^{2g} a_i = gw$ by our assumption.

$\square$

Proposition 11.5. *Let $\pi_f = \pi_{1,f}\times\ldots\times\pi_{r,f}$ be a representation of $G(\mathbb{A}_f)$ contributing to $Ind^{G(\mathbb{A}_f)}_{P_{I(J)}(\mathbb{A}_f)} H^{*,M_{I(J)}}_{2,disc}(V_{w_J(\chi+\rho)-\rho})$ for a fixed η-invariant J in the sense that for $i=1,\ldots,r$ we have an automorphic representation $\pi_i = \pi_{i\infty}\times\pi_{i,f}$ of $GL_{j_i}(\mathbb{A})$ in the discrete spectrum, and $\pi_i\otimes|\det|^{k_i/2}$ contributes to $H^{*,GL_{j_i}}_{2,disc}(V_{\chi_i})$. Assume $\eta(\pi_f)\cong\pi_f$.*

- *(a) We have $\eta(\pi_i)\cong\pi_i$ for $i=1,\ldots,r$. If $\pi_i = MW(\rho_i,n_i)$ with $j_i = m_i\cdot n_i$ and a cuspidal automorphic representation ρ_i of $GL_{m_i}(\mathbb{A})$ we have $\eta(\rho_i)\cong\rho_i$.*
- *(b) The central character $\omega_i = \omega(\pi_i) = \omega_{i\infty}\cdot\omega(\pi_{i,f})$ is a quadratic character.*
- *(c) If $j_i = 2\gamma_i+1$ is odd, then $\pi_i\otimes\omega_i$ is an η-invariant automorphic representation of $PGL_{2g_i+1}(\mathbb{A})$.*
- *(d) If $j_i = 2\gamma_i$ is even, then $\omega_{i\infty}(-1) = (-1)^{\gamma_i}$.*

Proof: (a) Let $\pi = \pi_1\times\ldots\times\pi_r$. By assumption π is a quotient of

$$\rho_1|\cdot|^{\frac{n_1-1}{2}}\times\ldots\times\rho_1|\cdot|^{\frac{1-n_1}{2}}\times\ldots\times\rho_r|\cdot|^{\frac{n_r-1}{2}}\times\ldots\times\rho_r|\cdot|^{\frac{1-n_r}{2}}.$$

Similarly $\eta(\pi)$ is a quotient of

$$\eta(\rho_r)|\cdot|^{\frac{n_r-1}{2}}\times\ldots\times\eta(\rho_r)|\cdot|^{\frac{1-n_r}{2}}\times\ldots\times\eta(\rho_1)|\cdot|^{\frac{n_1-1}{2}}\times\ldots\times\eta(\rho_1)|\cdot|^{\frac{1-n_1}{2}}.$$

By the extended version of the strong multiplicity one theorem [JS81b, Theorem 4.4] the assumption $\eta(\pi_f)\cong\pi_f$ implies that for every $i=1,\ldots r$ and every $\nu_i\le n_i-1$ there exists $j\le r$ and $\nu'_j\le n_j-1$ such that we have $\eta(\rho_i)|\cdot|^{\frac{n_i-1}{2}-\nu_i}\cong\rho_j|\cdot|^{\frac{n_j-1}{2}-\nu'_j}$. This implies $m_i=m_j$. But $H^p(\mathfrak{g}_{m_i},K_\infty^{m_i},\rho_{i,\infty}\otimes V_{\chi_i})\neq 0$ for some $p\ge 0$ implies

$$0\neq H^p(\mathfrak{g}_{m_i},K_\infty^{m_i},\eta(\rho_{i,\infty})\otimes{}^{\eta}V_{\chi_i}))=H^p(\mathfrak{g}_{m_i},K_\infty^{m_i},\eta(\rho_{i,\infty})\otimes(V_{\chi_i}\otimes\kappa_i\circ\det))$$

since ${}^{\eta}V_{\chi_i}=V_{\eta(\chi_i)}=V_{\chi_i}\otimes\kappa_i\circ\det$. This implies $0\neq H^p(\mathfrak{g}_{m_i},K_\infty^{m_i},(\eta(\rho_{i,\infty})\otimes|\det|^{k_i})\otimes V_{\chi_i})$. Similarly $H^{p'}(\mathfrak{g}_{m_i},K_\infty^{m_i},\rho_{j,\infty}\otimes V_{\chi_j})\neq 0$ for some $p'\ge 0$. But then the fact, that $\eta(\rho_i)$ and ρ_j differ by a power of the modulus character, implies that χ_i and χ_j have the same characteristic set. Since $S_{\chi_i}\neq S_{\chi_j}$ for $i\neq j$ we get $i=j$. Then the bijectivity of the map $(i,\nu_i)\mapsto(j,\nu'_j)$ implies immediately that we have $\nu'_j=0$ for $\nu_i=0$, and thus we get $\eta(\rho_i)\cong\rho_i$. From this we deduce $\eta(\pi_i)\cong\pi_i$.

(b) is a consequence of (a) and $\omega(\eta(\pi_i))=\omega(\pi_i)^{-1}$.

(c) We have $\omega(\pi_i\otimes\omega_i)=\omega(\pi_i)\cdot\omega_i^{2g_i+1}=\omega_i^{2(g_i+1)}=1$ by (b).

(d) Since π contributes to the cohomology we have some non vanishing class in $H^p(\mathfrak{g}_{j_i},K_\infty^{j_i},\pi_{i,\infty}\otimes V_\chi)$. Therefore we have some non vanishing K_∞-equivariant homomorphism from $\Lambda^p(\mathfrak{g}_{j_i}/\mathfrak{k}^{j_i})$ to $\pi_{i,\infty}\otimes V_\chi$. Since $-id\in\mathrm{SO}_{2\gamma_i}\subset K_\infty$ acts as identity on $\Lambda^p(\mathfrak{g}/\mathfrak{k})$, as $\omega_{i\infty}(-1)$ on $\pi_{i,\infty}$ and as $(-1)^{\gamma_i}$ on V_χ by 11.4 (since k_i is odd), the claim is clear.

□

(11.6) PROOF OF THE MAIN THEOREM 5.3.
If τ is an irreducible automorphic representation on $\mathrm{Sp}_{2g}(\mathbb{A})$ such that τ_f appears with non trivial multiplicity in the alternating sum

$$\sum_i(-1)^iH^i(\mathfrak{sp}_{2g},U_g,L_{2,disc}(\mathrm{Sp}_{2g}(\mathbb{Q})\backslash\mathrm{Sp}_{2g}(\mathbb{A}))\otimes V_\chi),$$

then theorem 10.15 tells us that there is a partition of S_χ into characteristic sets $\tilde S_i$ such that τ weakly lifts to a representation of the form $\pi=\pi_1\times\ldots\times\pi_r$ as in proposition 11.5 with $\eta(\pi_f)=\pi_f$. Here π_i are automorphic representations in the discrete spectrum, and by the main result of [MW89] they are of the form $\pi_i=MW(\rho_i,n_i)$ with positive integers n_i and cuspidal representations ρ_i on some GL_{m_i}. Then $\tilde S_i=MW(S_i,n_i)$ for some n_i-admissible characteristic set S_i of type X_{γ_i}, and the ρ_i are up to twist by a power of the modulus character cohomological with respect to some coefficient system V_{χ_i} with $S_i=S_{\chi_i}$. By 11.5(a) we have $\eta(\rho_i)=\rho_i$. By 11.5(b) we can write $\omega_{\rho_i}=\chi_{d_i}$ for some $d_i\in\mathbb{Q}^*/(\mathbb{Q}^*)^2$. In the case that S_i is of type C_{γ_i} we put $\pi^{(i)}=\rho_i\otimes\chi_{d_i}$. In all other cases we put $\pi^{(i)}=\rho_i$.

Again we have $\eta(\pi^{(i)}) = \pi^{(i)}$. By the main result of [Sou05] (compare [CPSS11, Thm. 6.1.]) we get that $\pi^{(i)}$ is a weak lift of some irreducible generic cuspidal automorphic representation $\pi_1^{(i)}$ on the group $\mathrm{Sp}_{2\gamma_i}$ in case that S_i is of type C_{γ_i}, respectively on either the group $G_1^{(i)} = \mathrm{SO}_{2\gamma_i+1}$ or on the group $G_1^{(i)} = \mathrm{SO}_{2\gamma_i}^{d_i}$. Since $\pi_1^{(i)}$ has a semi weak lift to the GL_* groups in question by [CPSS11] and since we have strong multiplicity one for $\mathrm{GL}_{2\gamma_i}$ we get, that $\pi^{(i)}$ is a semi weak lift of $\pi_1^{(i)}$. But now lemma 4.2 together with the description of $\mathcal{L}(\pi_\infty^{(i)})$ imply, that the group $G_1^{(i)}$ is of the correct type X_{γ_i}. The relation $\Pi_{i=1}^r d_i = 1$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is a consequence of $\Pi_{i=1}^r \chi_{d_i} = Pi_{i=1}^r \omega_i = \omega_\pi = 1$. The remaining assertions are obvious or may be deduced from 11.5(c) and (d). □

(11.7) Proof of the Converse Theorem 5.4: The proof consists in reversing the arguments of the proof of the Main Theorem. But one has to check that for each octupel satisfying the conditions the η-action on the corresponding part of the cohomology has a non trivial Lefschetz number: The cohomology is of the form

$$I = Ind_{P_I(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\ H^*(\mathfrak{m}_I, K_\infty \cap M_I(\mathbb{R}); (\pi_1 \times \ldots \pi_r) \otimes V_S)$$

for a suitable coefficient system V_S. We can calculate in GL_{2g+1}, such that $M_I = G^{(1)} \times \ldots \times G^{(r)}$ and such that $V_S = V_1 \otimes \ldots \otimes V_r$. We may assume $G^{(1)} = \mathrm{GL}_{2\gamma_1+1}$ (i.e. type C_{γ_1}), so that $G^{(i)} = \mathrm{GL}_{2\gamma_i}$ for $i \geq 2$. For $i \geq 2$ there are elements $k_i \in K_\infty \cap M_I(\mathbb{R})$ which are $-id$ in the first factor $G^{(1)}$, lie in $O_{2\gamma_i} - SO_{2\gamma_i}$ in the component i and are trivial in all other components. Then $K_\infty \cap M_I(\mathbb{R})$ is generated by the k_i and by its connected component $(K_\infty \cap M_I(\mathbb{R}))^\circ$. Let ω_1 be the archimedean component of the central character of π_1. This is a quadratic character on $\mathbb{R}^*$ with $\epsilon = \omega(-1) = sign(d_1) \in \{\pm 1\}$. Since the center of $G^{(1)}$ acts trivially on V_1 we get that equivariance under $K_\infty \cap M_I(\mathbb{R})$ translates into

$$I = Ind_{P_I(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\ \left(H^*(\mathfrak{g}^{(1)}, K_\infty^{(1)}, \pi_1 \otimes V_1) \times H^*(G^{(2)})^\epsilon \times \ldots \times H^*(G^{(r)})^\epsilon \right),$$

where $H^*(G^{(i)})^\epsilon = H^*(\mathfrak{g}^{(i)}, K_\infty^{(i)}, \pi_i \otimes V_i)^\epsilon$ are the $O_{2\gamma_i}/SO_{2\gamma_i}$ eigenspaces as in 4.11 for $i \geq 2$, and where now $K_\infty^{(i)}$ are connected subgroups. Then propositions 4.10 and 4.12 imply, that the Lefschetz number of $\eta \times h_f$ equals $\pm 2^{g-(r-1)}$ times the trace of ηh_f on $Ind_{P_I(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \pi_{1,f} \times \pi_{r,f}$. Since this is not identically zero the usual trace formula arguments give the claim.

□

Remark: The number $\pm 2^{g-(r-1)}$ should be the multiplicity, with which an individual $\mathrm{Sp}_{2g}(\mathbb{A}_f)$ module contributes to the alternating sum of the cohomology groups.

12 Examples and Complements

(12.1) Type C_1. For $k \geq 0$ the characteristic set $S = \{-k-1, 0, k+1\}$ is of type C_1. It corresponds to the coefficient system Sym^k on $G_1 = \mathrm{Sp}_2 = \mathrm{SL}_2$.

π_1 is cohomological with respect to S if $\pi_{1,\infty} \subset D(k+1)$. This means that π_1 corresponds to holomorphic (resp. antiholomorphic) cusp forms on $\mathrm{Sp}_2 = \mathrm{SL}_2$ of weight $k+2$. On $G = \mathrm{PGL}_3$ we get the adjoint lift $\pi = Ad(\pi_1)$ in the sense of [GJ78] (compare [Fl94] for a trace formula approach). We have $\pi_\infty = D(0, 2k+1) \times \sigma_{0,1}$. A cuspidal automorphic representations π_1 of $G_1(\mathbb{A})$ is a subrepresenations of some cuspidal automorphic representation $\tilde{\pi}_1$ of GL_2, which is unique up to character twists ([Ram00]), and we may write $\pi = Sym^2(\tilde{\pi}_1) \otimes \omega_{\tilde{\pi}_1}^{-1}$ (independent of the choice of $\tilde{\pi}_1$). π is irreducible cuspidal unless $\tilde{\pi}_1$ is a CM-form, i.e. unless $\tilde{\pi}_1$ is a Weil representation of a Größencharacter.

(12.2) TYPE D_1. For $k \geq 0$ the characteristic set $S = \{-k-1, k+1\}$ is of type D_1. Since $(-1)^1 d > 0$ we get an imaginary quadratic extension $F = \mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. The group $G_1 = \mathrm{SO}_2^d$ is the kernel of the norm map $Res_{F/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m$, so that $G_1(\mathbb{R}) = \{z \in \mathbb{C} | z\bar{z} = 1\}$. Then $\pi_{1,\infty}(z) = z^{k+1}$. The character $\theta : \mathbb{A}_F^*/F^* \to \mathbb{C}^*, a \mapsto \pi_1(a \cdot \bar{a}^{-1})$ is a Größencharacter with archimedean component $\theta_\infty(z) = (z/|z|)^{2k+2}$. Then $\pi = \mathcal{W}(\theta)$ is the Weil representation attached to this Größencharacter. We have $\pi_\infty = D(0, 2k+1)$. Since $\theta(a) = 1$ for $a \in \mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ we get $\omega_\pi = \chi_d$.

We remark that

(12.3) TYPE B_1. For $k \geq 0$ the characteristic set $S = \left\{-k - \frac{1}{2}, k + \frac{1}{2}\right\}$ is of type B_1. The elementary pairs are cuspidal automorphic representations π_1 of $G_1 = \mathrm{SO}_3 \cong \mathrm{PGL}_2$, which may be viewed as representations π of $G = \mathrm{GL}_2$. They are cohomological with respect to S if $\pi_\infty = D(0, 2k+1)$, and then they may be viewed as classical cusp forms of weight $2k+2$ with trivial multiplier.

(12.4) TYPE D_2. For $a \geq b \geq 0$ the characteristic set

$$S = \{-a-2, -b-1, b+1, a+2\}$$

of type D_2 gives rise to a coefficient system, which pulls back to the system $Sym^{a+b+2} \otimes Sym^{a-b}$ under the exceptional isogeny $i : \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ (for the coefficient systems one can consider this over $\bar{\mathbb{Q}}$). In the case $d = 1$ we can write $i^*(\pi_1) = \sigma_1 \otimes \sigma_2$, where the cuspidal representations σ_i of $\mathrm{SL}_2(\mathbb{A})$ are contained in cuspidal representations $\tilde{\sigma}_i$ of $\mathrm{GL}_2(\mathbb{A})$ and we may assume $\omega_{\tilde{\sigma}_1} \cdot \omega_{\tilde{\sigma}_2} = 1$. Then the automorphic representation π on $\mathrm{GL}_4(\mathbb{A})$ is of the form $\pi = \tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2$ in the sense of [Ram00]. We have $\sigma_{1,\infty} = D(a+b+3)$ and $\sigma_{2,\infty} = D(a-b+1)$ and thus π_1 corresponds to a pair of classical cusp forms (f, g) of weights $a+b+4$ and $a-b+2$ respectively. In view of the different archimedean types $\tilde{\sigma}_2$ is not a twist of $\tilde{\sigma}_1$ and therefore $\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2$ is a cuspidal representation of GL_4 unless $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are both CM-representations with respect to the same imaginary quadratic extension $E/\mathbb{Q}$. This is a consequence of the cuspidality criterion of [Ram00, ch. 3]: if $\tilde{\sigma}_1 = \mathcal{W}(\theta)$ for a Größencharacter θ on $\mathbb{A}_E$, but $\tilde{\sigma}_2$ has no CM by E, then the base change $\tilde{\sigma}_{2,E}$ of $\tilde{\sigma}_2$ is cuspidal and it admits no self twist with $\bar{\theta} \cdot \theta^{-1}$, since this latter character has infinite order.

In the case $d \neq 1$ we have $\mathrm{SO}_4^d \cong Res_{F/\mathbb{Q}}\mathrm{SL}_2/\{\pm 1\}$ with $F = \mathbb{Q}(\sqrt{d})$ real quadratic, so that π_1 corresponds to a Hilbert modular form of weight $(a+b+4, a-b+2)$.

In the case $a = b \geq 0$ we can write

$$\{-a-2, -a-1, a+1, a+2\} = MW(\left\{-a-\frac{3}{2}, a+\frac{3}{2}\right\}, 2)$$

and we get classes in the discrete spectrum, which are described by a holomorphic cusp form of weight $2a+4$ and the residue of the weight 2 Eisenstein series.

(12.5) The Case $g = 2$. For $a \geq b \geq 0$ consider the characteristic set of type C_2:

$$S \quad = \quad \{-a-2, -b-1, 0, b+1, a+2\}\,.$$

(a) The trivial partition $S = S$ corresponds to cuspidal representations of $\mathrm{Sp}_4(\mathbb{A})$, which lift to irreducible cuspidal representations of $\mathrm{PGL}_5(\mathbb{A})$.

(a') In the case $a = b = 0$ we can write $S = MW(\{0\}, 5)$ (where $\{0\}$ is of type C_0) and get the one dimensional representations in the residual spectrum on G_1 and on G.

(b) The partition $S = \{0\} \cup S_2$, where $S_2 = \{-a-2, -b-1, b+1, a+2\}$ is of type D_2, corresponds to cuspidal automorphic representations τ of $\mathrm{Sp}_4(\mathbb{A})$, which are endoscopic lifts of representations π_1 of SO_4^d with $d > 0$. In case $d = 1$ the representation π_1 is the restriction of a cuspidal representation $\tilde{\sigma}_1 \times \tilde{\sigma}_2$ of $\mathrm{GL}_2 \times \mathrm{GL}_2/\mathbb{G}_m = \mathrm{GSO}_4$ to SO_4, where $\omega_{\tilde{\sigma}_1} \cdot \omega_{\tilde{\sigma}_2} = 1$. Then $\pi = 1 \times (\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2)$ on $\mathrm{PGL}_5(\mathbb{A})$. Here $\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2$ is cuspidal, if $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are not CM-forms for the same imaginary quadratic extension of $\mathbb{Q}$. τ is the restriction to Sp_4 of an endoscopic lift of $\tilde{\sigma}_1 \times \tilde{\sigma}_2$ to GSp_4 in the sense of [Wei09]. If $d \neq 1$ then τ is an endoscopic lift to Sp_4 of a cuspidal representation π_1 on the quasisplit $\mathrm{SO}_4^d \cong Res_{F/\mathbb{Q}}\mathrm{SL}_2/\{\pm 1\}$, but not a restriction of an endoscopic representations of GSp_4.

(b') For $a = b$ one can write $S = \{0\} \cup MW(\{-a-\frac{3}{2}, a+\frac{3}{2}\}, 2)$ and gets $\pi = 1 \times MW(\pi^{(2)}, 2)$, which lies in the residual spectrum on the GL_4 factor. Then τ is either a CAP-representation with respect to the Siegel parabolic i.e. of Saito-Kurokawa type [PS83] or it is a residual representation of Sp_4, whose contribution to cohomology is described in the work of Schwermer [Schw86], [Schw95, Prop. 4.6.(2)]. This case may be viewed as a version of (b), where $\tilde{\sigma}_1$ is a one dimensional (residual) representation.

(c) The partition $S = S_1 \cup S_2$ with $S_1 = \{-a-2, 0, a+2\}$ of type C_1 and $S_2 = \{-b-1, b+1\}$ of type D_1 corresponds to τ on Sp_4, which are restrictions of automorphic representations $\tilde{\tau}$ of GSp_4, whose associated rank 4 motive is a tensor product of two GL_2 motives, one of them being CM: let $\pi_1^{(1)}$ be a cuspidal representation of GL_2, which is not of CM type, with $\pi_{1,\infty}^{(1)} = D(a+2, 0)$ and let $\pi^{(1)}$ be its adjoint lift to a cuspidal representation of PGL_3. Let $\sigma = \pi_1^{(2)}$ be an

automorphic character on the norm 1 group of an imaginary quadratic extension $F = \mathbb{Q}(\sqrt{d})/\mathbb{Q}$ with $\sigma_\infty(z) = z^{b+1}$ and let $\pi^{(2)} = \mathcal{W}(\theta)$ be the cuspidal Weil representation of GL_2, where $\theta(x) = \sigma(x \cdot \bar{x}^{-1})$. Then τ lifts to $(\pi^{(1)} \otimes \chi_d) \times \mathcal{W}(\theta)$ on PGL_5. If one extends σ to a Grossencharacter $\tilde{\sigma}$ of $\mathbb{A}_F^*/F^*$, one can arrange $\tilde{\tau}$ on $\mathrm{GSp}_4 = \mathrm{GSpin}_5$ in such a way that it lifts to $\pi_1^{(1)} \boxtimes \mathcal{W}(\sigma) \otimes \chi$ on $\mathrm{GL}_4 \times \mathbb{G}_m$, where χ is the product of the central character of $\pi_1^{(1)}$ and of the restriction of $\tilde{\sigma}$ to $\mathbb{A}_Q^*/\mathbb{Q}^*$. In special cases these representations have been constructed by Ramakrishnan and Shahidi [RS07, Theorem B'].

(d) The partition $S = S_1 \cup S_2$ with $S_1 = \{-b-1, 0, b+1\}$ of type C_1 and $S_2 = \{-a-2, a+2\}$ of type D_1 is completely analogous to the case (c), but with different weights.

(d') If $b = 0$ in case (d) we may write $S_1 = MW(\{0\}, 3)$. Then τ is a CAP-representations with respect to the Klingen parabolic ([Sou88]).

(e) The partition $S = \{0\} \cup \{-a-2, a+2\} \cup \{-b-1, b+1\}$ may be viewed as a degenerate case of the preceeding cases. If one has $\sigma_1 = \mathcal{W}(\theta_1)$ and $\sigma_2 = \mathcal{W}(\theta_2)$ in case (b), where θ_1 and θ_2 are Größencharacters on $\mathbb{A}_F^*/F^*$ for the same imaginary quadratic extension $F/\mathbb{Q}$, such that $\theta_1\theta_2$ is trivial on $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ then we have $\sigma_1 \boxtimes \sigma_2 = \mathcal{W}(\theta_1\theta_2) \times \mathcal{W}(\theta_1\bar{\theta}_2)$. If $\pi_1^{(1)}$ in case (c) or (d) is a Weil representation, we get another way to interpret this degeneracy.

(12.6) The case $g = 3$. Let $a \geq b \geq c \geq 0$ be integers. Then we get 3 partitions of the characteristic set

$$S = \{-a-3, -b-2, -c-1, 0, c+1, b+2, a+3\}$$

of type C_3 into a characteristic set S_1 of type C_1 and a set S_2 of type D_2. We get the following weights of classical modular forms (compare [BFG11, 7.10.]), which can be lifted to Siegel modular forms:

$$\begin{aligned}
S_1 = \{-a-3, 0, a+3\} &\text{ of weight } a+4, &&\text{and } S_2 \text{ of weights } b+c+4,\ b-c+2\\
S_1 = \{-b-2, 0, b+2\} &\text{ of weight } b+3, &&\text{and } S_2 \text{ of weights } a+c+5,\ a-c+3\\
S_1 = \{-c-1, 0, c+1\} &\text{ of weight } c+2, &&\text{and } S_2 \text{ of weights } a+b+6,\ a-b+2.
\end{aligned}$$

(12.7) The Ikeda Lift. For $k \geq 0$ classical scalar valued holomorphic Siegel modular forms of weight $k+g+1$ on Sp_{2g} (i.e. with automorphy factor $det(CZ+D)^{k+g+1}$) correspond to cohomology classes with the coefficient system $(k, \ldots, k, -k, \ldots, -k) \in X^*(T_H) = \mathbb{Z}^g$. The characteristic set (of type C_g) is

$$S = \{-k-g, \ldots, -k-1, 0, k+1, \ldots, k+g\}.$$

If $g = 2\gamma$ is even then we have a partition $S = \{0\} \cup MW(S_1, 2\gamma)$, where $S_1 = \left\{-k-\gamma-\frac{1}{2}, k+\gamma+\frac{1}{2}\right\}$ corresponds to classical holomorphic forms of weight

$2k + 2\gamma + 2$. This gives an interpretation of the Ikeda-Lift [Ik01][KoKo05], and in the special case $\gamma = 1$ we again get the Saito-Kurokawa-lift [PS83].

(12.8) Relations with elliptic endoscopy of GSp_{2g}. Next we try to understand the relation of theorems 5.3, 5.4 with the work of Morel [Mor11]: If $S = \bigcup_{i=1}^r MW(S_i, n_i)$ is a decomposition and $d_i \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ a family as in theorem 5.3, let $I_0 = \{1, \ldots, r\}$, assume S_1 is of type C_{γ_1} and let $(\pi^{(i)}, \pi_1^{(i)})_{i \in I_0}$ be a family of elementary particles, coming from an automorphic representation τ of $\mathrm{Sp}_{2g}(\mathbb{A})$.
The set Σ of all subsets I of I_0 with $1 \notin I$ is a group under the symmetric difference Δ with neutral element $\emptyset$. The kernel of the homomorphism $\pi : \Sigma \to \mathbb{Q}^*/(\mathbb{Q}^*)^2,\ I \mapsto \prod_{i \in I} d_i$ is thus

$$\Sigma_{\mathbb{Q}} \quad = \quad \{I \subset I_0 | 1 \notin I,\ \Pi_{i \in I} d_i = 1\}\,.$$

For $I \in \Sigma_{\mathbb{Q}}$ we can put $S_I = \bigcup_{i \notin I} MW(S_i, n_i)$ and $S'_I = \bigcup_{i \in I} MW(S_i, n_i)$. Then S_I is of type C_{g_1} and S'_I is either empty or of type D_{g-g_1} if $g_1 < g$. It is known that the elliptic endoscopic groups of GSp_{2g} are of the form $G_{g_1} = G(\mathrm{Sp}_{2g_1} \times \mathrm{SO}_{2(g-g_1)})$ ([Mor11, Prop. 2.1.1]). Now for each $I \in \Sigma_{\mathbb{Q}}$ their exists by 5.4 an automorphic representation τ_I on $\mathrm{Sp}_{2g_1}(\mathbb{A})$, such that τ_I lifts to $\Pi_{i \notin I} MW(\pi^{(i)}, n_i)$ and their should exist (comp. [Ar12]) an automorphic representation τ'_I on $\mathrm{SO}_{2(g-g_1)}(\mathbb{A})$, which lifts to $\Pi_{i \in I} MW(\pi^{(i)}, n_i)$. Then $\tau_I \otimes \tau'_I$ is a subrepresentation of some π_I on $G_{g_1}(\mathbb{A})$, which should be related to τ in the sense of elliptic endoscopy. We remark that $g - g_1$ is even, since $d_i < 0$ if S_i is of type D_{γ_i} with odd γ_i.

Let $D \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2$ be the image of π and let $E := \mathbb{Q}(\sqrt{D}) := \mathbb{Q}(\sqrt{d_2}, \ldots, \sqrt{d_r})$ be the corresponding field extension of degree $\#(D)$. The Galois group $\Gamma_{E/\mathbb{Q}} = \Gamma_{\mathbb{Q}}/\Gamma_E$ is canonical dual to D in the category of $\mathbb{F}_2$ vector spaces: $\sigma(d) = \sigma(\sqrt{d})/\sqrt{d} \in \{\pm 1\}$. Now Σ is self dual with respect to the bilinear form $\beta(I, J) = (-1)^{\#(I \cap J)}$ and the canonical map $\pi^\vee : \sigma \mapsto \left\{i \in I_0 | i \neq 1 \text{ and } \sigma(\sqrt{d_i}) = -\sqrt{d_i}\right\}$ is dual to π with respect to these pairings.

One expects for each $i \in I_0$ the existence of a λ-adic representation V_i of $\Gamma_{\mathbb{Q}}$ of dimension 2^{γ_i} such that the Frobenius eigenvalues are related to the Satake parameters of the automorphic representation $\pi_1^{(i)}$. For $i \geq 2$ the restriction of this representation to Γ_{E_i} where $E_i = \mathbb{Q}(\sqrt{d_i})$ should split into two representations V_i^+ and V_i^- of dimension $2^{\gamma_i - 1}$ in such a way that we have $V_i = Ind_{\Gamma_{E_i}}^{\Gamma_{\mathbb{Q}}} V_i^+ = Ind_{\Gamma_{E_i}}^{\Gamma_{\mathbb{Q}}} V_i^-$ if $d_i \neq 1$. For $I \in \Sigma$ we can form

$$V_I \;=\; Ind_{\Gamma_E}^{\Gamma_{\mathbb{Q}}} \,(V_1 \otimes V_2^{\epsilon_2} \otimes \ldots \otimes V_r^{\epsilon_r}),$$

where $\epsilon_i = -$ for $i \in I$ and $\epsilon_i = +$ for $i \notin I$. These are representations of dimension $2^g/2^{r-1} \cdot [E : \mathbb{Q}] = 2^g/\#(\Sigma_{\mathbb{Q}})$. Then we have $V_I \cong V_{I'}$ if $I \equiv I' \mod \pi^\vee(\Gamma_E)$ and finally $V_1 \otimes \ldots \otimes V_r \;=\; \bigoplus_{I \in \Sigma/\pi^\vee(\Gamma_E)} V_I$. Here the index set $\Sigma/\pi^\vee(\Gamma_E)$ is canonical dual to $\Sigma_{\mathbb{Q}} = \ker(\pi : \Sigma \to D)$. Now τ should be associated to one of the λ-adic representations Σ_I, and the τ' in the packet of τ may correspond to $V_{I'}$ for different I'.

(12.9) In the notations of the example 12.6 ($g = 3$) let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ with $S_1 = \{0\}$, $S_2 = \{a + 3, -a - 3\}$, $S_3 = \{b + 2, -b - 2\}$, $S_4 = \{c + 1, -c - 1\}$ and $d_1 = d_2 = d_3 = d_4 < 0$. Then we have $\Sigma = \{\emptyset, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, which means that we have to combine two elementary particles to get an elliptic endoscopic representation. This should lead to motives of rank 2 in the cohomology of the associated Shimura variety, but this phenomenon does not appear in the totally unramified case considered in [BFG11].

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